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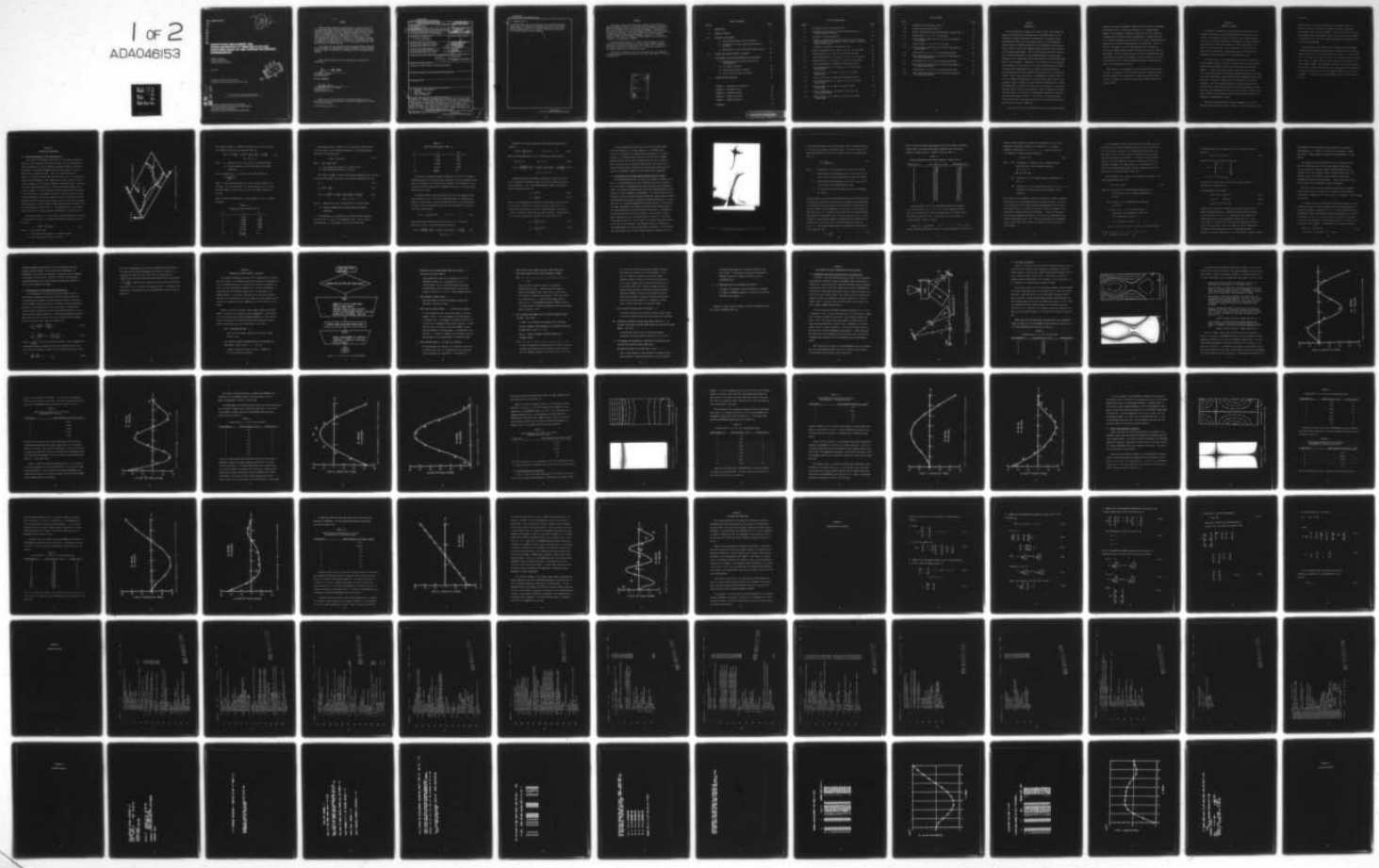

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QUANTITATIVE DISPLACEMENT AND STRAIN DISTRIBUTION OF VIBRATING PLATE-LIKE STRUCTURES BASED ON TIME AVERAGE HOLOGRAPHIC INTERFEROMETRY

JAMES C. MacBAIN
PROPULSION BRANCH
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JULY 1977

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A technique for computing the bending strain resulting from the resonant modal deformation of a vibrating plate-like structure is described. Interferometric fringes obtained by time-average holography are used as the basis for generating a mathematically continuous series approximation of the structure's out-of-plane displacement. The terms of the series consist of the eigenfunctions of a simple beam having either clamped-free or free-free boundary conditions. Use is made of a linear least squares equation solver to compute the coefficients

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20. ABSTRACT (Cont'd)

of the displacement series terms based upon the experimental holographic fringe values. The technique is programmed for use on an interactive computer terminal. The method is tested by comparing it to the results of an eigenvalue analysis of a cantilevered plate analyzed using the finite element computer program, NASTRAN.

FOREWORD

This report covers work carried out at AFAPL's Turbo Structures Research Laboratory (TSRL) on the quantitative determination of displacements and bending strains resulting from resonant modal deformation of plate-like structures such as turbomachinery blading. The objective of the research effort was to develop a workable technique for efficiently obtaining surface bending strain information from the interferometric fringes produced by time average holographic testing of a vibrating structure. This type of information relates directly to the structural integrity of turbomachinery blading.

The work was performed in the Turbine Engine Division of the Air Force Aero Propulsion Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, under Project 3066, Task 12, Work Unit 21 and Project 2307, Task S2, Work Unit 01. The effort was conducted by Dr. James C. MacBain of the Propulsion Branch.

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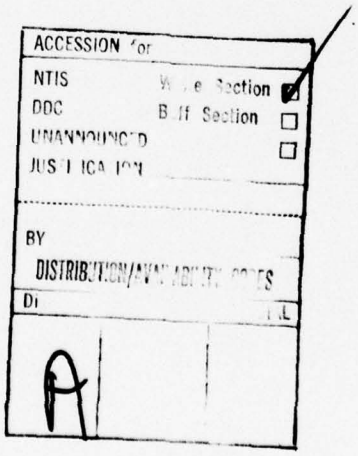

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SECTION I

INTRODUCTION

Since its discovery by Powell and Stetson in 1965, time average holographic interferometry has developed into an extremely useful tool for studying the resonant mode shapes and frequencies of vibrating bodies. The method has proven to be particularly useful in the turbine engine industry where the vibration properties of turbomachinery blading must be known in order to insure that dangerous blade resonances do not coincide with operating engine speeds. While the mode shape and natural frequency can be directly determined by time average holographic interferometry, knowledge of the corresponding strain distribution for each vibration mode shape is also desirable. Computation of the surface strain has presented some problems in the past because of the error magnification that results in using inexact experimental data to compute the second derivative. Various approaches have been proposed in the literature for attacking this problem. Waters, Aas, and Erf (Ref. 1) determined the bending strain of the first bending vibration mode of a turbine blade using a "smooth curve" approach in order to smooth out errors caused by differentiation. Taylor and Brandt (Ref. 2) carried out a very good error analysis of strain computations based on cubic spline functions. Recently, Dandliker, Ineichen, and Mastner (Ref. 3) reported on an experimental technique for measuring the interference phase to within .3° at any point on the displacement fringe pattern; thus, improving the accuracy of the holographic displacement data by two orders of magnitude.

In the vein of the first two references cited, the present study uses

standard time average holographic fringes as the raw data for determining the bending strain of a plate-like structure, e.g., turbine engine blading. The holographic fringes are used as the raw data to generate a mathematically continuous series approximation of a plate-like structure's normal displacement where the terms of the series approximation are clamped-free or free-free eigenfunctions of a simple beam. Since the series is continuous, it can be differentiated twice to obtain the curvature or bending strain resulting from one of the plate's vibration mode shapes. The use of beam functions in the series approximation has the advantage that the plate's geometric boundary conditions are identically satisfied and its natural boundary conditions, satisfying equilibrium, are approximately satisfied.

What follows is a development of the theory upon which the method is based, a description of the resulting interactive computer program, and some examples of its application to some plate vibration modes. The results are compared with bending strain values obtained using finite element analysis.

SECTION II

SUMMARY OF RESULTS

A technique for computing the bending strain resulting from the resonant modal deformation of vibrating plate-like structures is described. Interferometric fringes obtained by time average holography are used as the basis for generating a mathematically continuous series approximation of a plate-like structure's normal displacement. The terms of the series consist of the clamped-free or free-free eigenfunctions of a simple beam. The bending strain is then obtained by computing the second derivative of the displacement series.

The coefficients of the displacement series terms are computed for a given segment (length or width) of a cantilevered plate-like structure based upon the holographic fringe values lying along the same plate segment. A linear least squares solution routine is used to solve for p series coefficients, called modal weighting coefficients, in terms of k normal displacement values obtained from k holographic fringe values where $k \geq p$. A "best fit" solution is thus obtained for the plate displacement. This least squares approach in conjunction with the fact the beam series functions exactly satisfy the plate's geometric boundary conditions and approximately satisfy the plate's natural boundary conditions, results in a displacement series that yields quite accurate displacement and bending strain values.

The technique described above has been programmed for use on a Tektronix 4010 interactive computer terminal. To use the program, called

HOLOCURVE, the user specifies the particular chordwise (free-free boundary conditions) or spanwise (clamped-free boundary conditions) plate segment to be studied, the number of terms to be used in the series approximation, and the holographic fringe values along the plate segment. A listing and plot of the plate's normal displacement and bending strain is generated by HOLOCURVE.

The accuracy and effectiveness of the technique used by HOLOCURVE is checked by determining the displacement and bending strain at selected segment locations for three different modes of vibration of a cantilevered plate. The results are compared to those of an eigenvalue analysis carried out on a finite element model of the plate using the finite element computer program, NASTRAN. The HOLOCURVE results are in excellent agreement with the finite element analysis except for cases where the bending strain is essentially zero as in the case of chordwise segment for a torsional mode shape. In cases such as this, HOLOCURVE issues a warning message to the user.

SECTION III

THEORETICAL DEVELOPMENT

3.1 Series Approximation of the Plate Deformation

In the spirit of Rayleigh's method (Ref. 4), introduced one hundred years ago in his treatise, "The Theory of Sound", one can approximate the normal deformation of a freely vibrating rectangular plate by the natural mode shapes of beams whose boundary conditions are similar to those of a plate along its respective edges. Thus, for a cantilevered plate (Figure 3.1), having one edge fixed and three edges free, one can model the plate's modal deformation in the clamped-free direction (parallel to the x axis in Figure 3.1) by a similar clamped-free beam mode shape. Similarly, the plate's modal deformation in its free-free direction (parallel to the y axis in Figure 3.1) can be modeled by suitably chosen natural mode shapes of a free-free beam. The boundary conditions of the plate and the corresponding clamped-free and free-free beams are similar in the sense that the geometric boundary conditions (deflections, rotation, and slope) are exactly satisfied while the natural boundary conditions (equilibrium conditions) are only approximately satisfied (Ref. 5). The ramifications of only approximately satisfying the natural boundary conditions will be discussed later.

Referring to Figure 3.1, let the plate's normal deflection with respect to the spanwise direction, x, (clamped-free boundary conditions) be represented by

$$W_x(x) = \sum_{i=1}^{p} A_i X_i(x) \tag{3.1}$$

where i - mode shape number

X_i - i-th natural mode shape for a clamped-free beam

A_i - modal weighting coefficient for mode i.

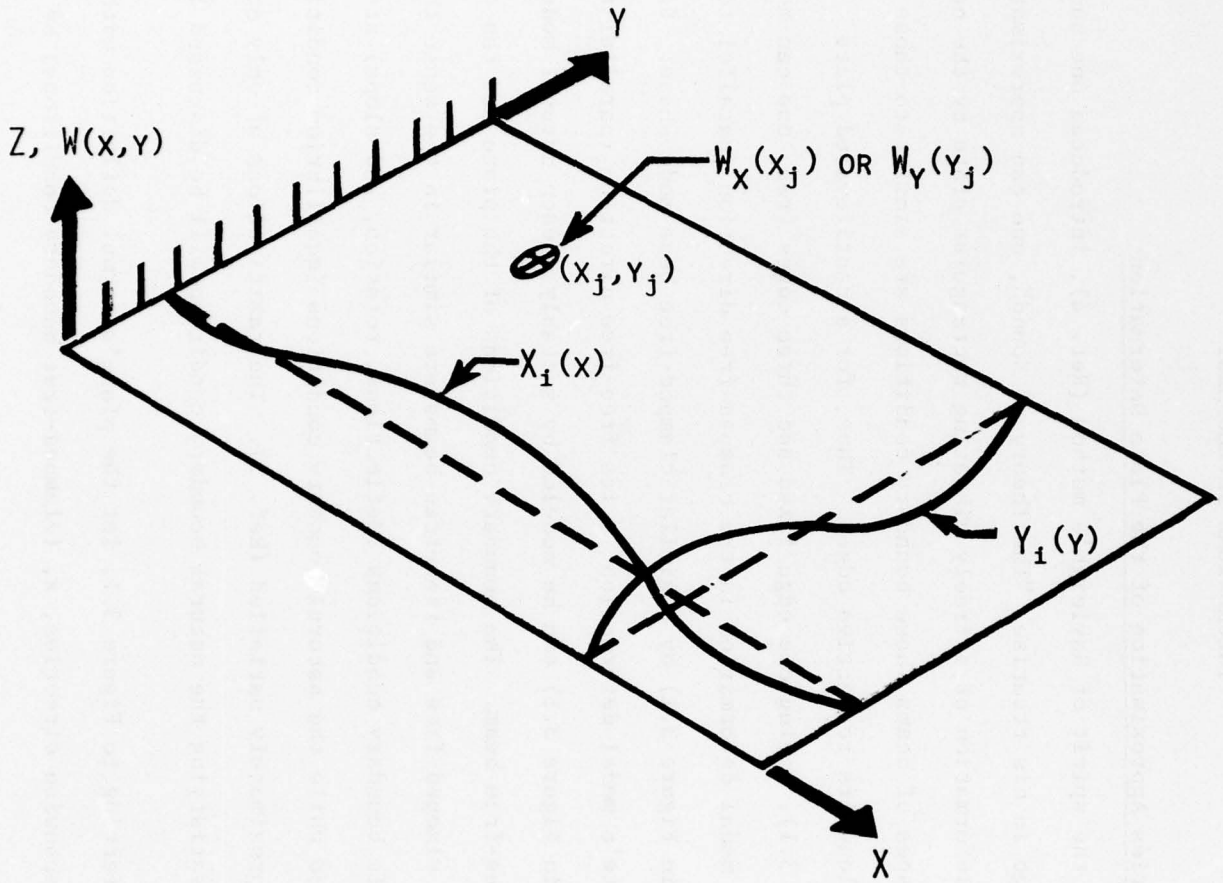


Figure 3.1 - Coordinate System for Cantilever Plate

For a beam of length, a, clamped at x=0 and free at x=a, the natural mode shapes are given by the expression (Ref. 6):

$$X_i(x) = \cosh\frac{\lambda_i x}{a} - \cos\frac{\lambda_i x}{a} - \alpha_i\left[\sinh\frac{\lambda_i x}{a} - \sin\frac{\lambda_i x}{a}\right] \qquad (3.2)$$

for i=1, 2, 3 ....

where λ_i - eigenvalue for the i-th mode of a clamped-free beam

α_i - constant dependent upon the mode number and boundary conditions.

The i-th eigenvalue, λ_i, is related to the beam properties by

$$\lambda_i = \left(\frac{\rho A L^4}{EI}\right)^{\frac{1}{2}} \omega_i$$

where ρ is the beam's mass density, A is the cross-sectional area of the beam, L is the beam length, E is Young's Modulus, I is the cross-sectional moment of inertia and ω_i is the i-th natural frequency of the beam.

Table 3.1 gives the eigenvalues, λ_i, and constants, α_i, for a clamped-free beam.

TABLE 3.1

Clamped-Free Beam Constants (Ref. 6)

i	λ_i	α_i
1	1.8751	.7340
2	4.6940	1.0184
3	7.8547	.9992
4	10.9955	1.0000
5	14.1371	.9999
6	17.2787	1.0
i>6	$\left(\frac{2i-1}{2}\right)\pi$	1.0

Referring once more to Figure 3.1, let the plate's normal deflection with respect to the chordwise direction, y, (free-free boundary conditions) be represented by:

$$W_y(y) = \sum_{i=1}^{q} B_i Y_i(y) \tag{3.3}$$

where i - mode shape number

Y_i - i-th natural mode shape for a free-free beam

B_i - modal weighting coefficient for mode i

For a beam of length, b, having free boundary conditions at y=0 and y=b, the natural mode shapes are given by the following expressions (Ref. 6):

$$Y_1 = 1 \tag{3.4}$$

$$Y_2 = \sqrt{3}\,(1 - \frac{2y}{b}) \tag{3.5}$$

$$Y_i(y) = \cosh\frac{\gamma_i y}{b} + \cos\frac{\gamma_i y}{b} - \beta_i\left[\sinh\frac{\gamma_i y}{b} + \sin\frac{\gamma_i y}{b}\right] \tag{3.6}$$

for i = 3, 4, 5 ...

where γ_i - eigenvalue for the i-th mode shape of a free-free beam

β_i - constant dependent upon the mode number and boundary conditions.

The eigenvalue, γ_i, is related to the free-free beam properties in the same way as λ_i was for a clamped-free beam. Table 3.2 gives the eigenvalues, γ_i, and constant, β_i, for a free-free beam.

TABLE 3.2

Free-Free Beam Constants (Ref. 6)

i	γ_i	β_i
3	4.7300	.9825
4	7.8532	1.0007
5	10.9956	.9999
6	14.1371	1.0000
$N \geq 7$	$\frac{(2N-3)\pi}{2}$	1.0

Note that the expressions given by equations (3.4) and (3.5) represent rigid body modes of the free-free beam. Equation (3.4) is a normalized translation of the beam in the z direction and equation (3.5) represents a rotation of the free-free beam about its midpoint, the vector of rotation being parallel to the x axis.

The modal functions given by equation (3.2) for the clamped-free beam and by equations (3.4) - (3.6) for the free-free beam are continuous with respect to each function's spatial variable and, hence, can be differentiated twice. Thus, from equation (3.1), the plate curvature in the x direction can be represented by

$$W_x''(x) = \sum_{i=1}^{p} A_i X_i''(x), \qquad i = 1, 2, 3, \ldots p \tag{3.7}$$

where primes denote differentiation with respect to x. The second derivative of the i-th clamped-free mode is given by

$$X_i''(x) = \left(\frac{\lambda_i}{a}\right)^2 \left[\cosh\frac{\lambda_i x}{a} + \cos\frac{\lambda_i x}{a} - \alpha_i\left(\sinh\frac{\lambda_i x}{a} + \sin\frac{\lambda_i x}{a}\right)\right] \tag{3.8}$$

for i=1, 2, 3 ...

Similarly, the plate curvature in the y direction can be represented by

$$W_y''(y) = \sum_{i=1}^{q} B_i Y_i''(y), \qquad i = 1, 2, 3, \ldots q \tag{3.9}$$

where the second derivative of the i-th free-free mode is given by

$$Y_i''(y) = 0, \qquad \text{for } i = 1, 2 \tag{3.10}$$

$$Y_i''(y) = \left(\frac{\gamma_i}{b}\right)^2 \left[\cosh\frac{\gamma_i y}{b} - \cos\frac{\gamma_i y}{b} - \beta_i\left(\sinh\frac{\gamma_i y}{b} - \sin\frac{\gamma_i y}{b}\right)\right] \tag{3.11}$$

$$\text{for } i = 3, 4, 5, \ldots$$

Neglecting in-plane stretching, equations (3.7) and (3.9) can be used to determine the bending strains in the plate caused by the out of plane deformation. For a plate undergoing pure bending, the strains can be obtained from (Ref. 5):

$$\varepsilon_x = -zW_x''(x) \tag{3.12}$$

$$\varepsilon_y = -zW_y''(y) \tag{3.13}$$

where z is the perpendicular distance from the plate's neutral surface to the point of interest on the plate's surface. Note that for a twisted plate-like structure such as a compressor blade, the distance, z, does not necessarily equal half the plate thickness. Using equations (3.7) and (3.9), equations (3.12) and (3.13) can be written as:

$$\varepsilon_x = -z \sum_{i=1}^{p} A_i X_i''(x) \tag{3.14}$$

$$\varepsilon_y = -z \sum_{i=1}^{q} B_i Y_i''(y) \tag{3.15}$$

Looking at equations (3.14) and (3.15), we see that the normal strain with respect to x (y held constant, i.e., constant chord location for a blade) or the normal strain with respect to y (x held constant, i.e., constant span location for a blade) can be computed provided that one can arrive at suitable values for the A_i's and B_i's. What follows is a way of accurately computing the A_i's and B_i's for a specified natural mode of vibration of a plate-like structure by utilizing the interference fringes from time average laser holography.

3.2 Determination of the Model Weighting Coefficients, A_i and B_i

The weighting coefficients, A_i and B_i, that appear in equations (3.1) and (3.3) for approximating the plate deformation and also in equations (3.7) and (3.9) for approximating the plate curvature can be determined using a least squares technique based on discrete displacement data points obtained by laser holographic interferometry. For a plate-like structure vibrating in one of its natural modes of vibration, time average holographic interferometry can be used to determine the modal deformation of the structure (Ref. 7). The modal deformation is characterized by a "contour map" of fringes appearing on the surface of the plate-like structure when the time average hologram is reconstructed with laser light. A photograph of one such reconstruction is shown in Figure 3.2 which shows a turbine blade vibrating in one of its natural modes of vibration. The fringes on the blade surface are a contour map of the blade's displacement normal to the plane of the hologram. The brightest fringes are nodal lines (zero displacement) for the particular mode of vibration. As one travels away from a nodal line, the dark fringes represent increasing values of

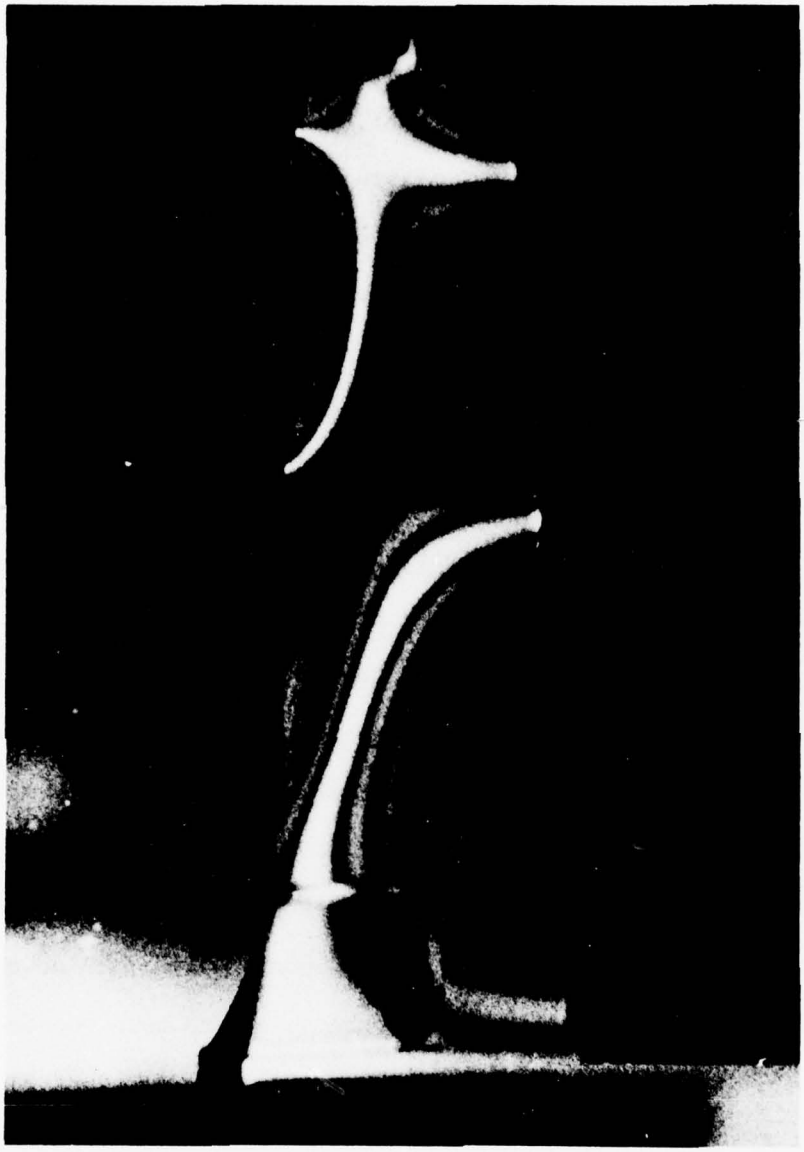


Figure 3.2 - Holographic Interferogram of a Turbine Blade Vibrating in a Torsional Mode at 25,943 Hz

the normal displacement and are approximately equal to integer multiples of one-quarter wavelength of the laser light. The actual value of the normal displacement of the i-th dark fringe is given by the expression (Ref. 7):

$$W_i = \frac{\Lambda\ \Omega_i}{2\pi(\text{Cos}\ \theta_1 + \theta_2)} \tag{3.16}$$

where Λ - wavelength of the laser light used to make the hologram

θ_1 - angle between the displacement vector of the plate and the line of sight of the observer/camera through the hologram

θ_2 - angle between the displacement vector of the plate and the laser "object" beam illuminating the plate

Ω_i - i-th root of the zero order Bessel function corresponding to fringe i.

If the specimen to be tested is positioned such that its principal plane of vibration is parallel to the plane of the hologram, and, in addition, is located such that its distance from the hologram is greater than five times its largest in-plane dimension, the value of θ_1, can be made very small and Cos θ_1, $\doteq$ 1. Further, if the object beam for illuminating the test specimen is placed very close to the hologram and the 5 to 1 ratio (distance of specimen from hologram to maximum specimen dimension) mentioned above is in effect, then θ_2 can also be made very small and Cos $\theta_2 \doteq 1$. With these criteria placed on experimental setup, equation (3.16) takes the form

$$W_i = \frac{\Lambda\ \Omega_i}{4\pi} \tag{3.17}$$

Table 3.3 gives the normal displacement for the first twenty interference fringes based on the above simplifying assumptions and a laser wavelength for a HeNe laser of $\Lambda = 6328\mathring{A} = 24.914$ μin (Ref. 8).

TABLE 3.3

Normal Displacement Values from Holographic Fringes (Ref. 8)

Fringe Order, i	i-th Root of J_o (Ω_i)	Displacement (μin)
1	2.405	4.768
2	5.520	10.94
3	8.634	17.12
4	11.79	23.37
5	14.93	29.60
6	18.07	35.82
6	21.21	42.06
8	24.35	48.29
9	27.49	54.51
10	30.64	60.74
11	33.78	66.97
12	36.92	73.20
13	40.06	79.43
14	43.20	85.66
15	46.34	91.89
16	48.48	98.11
17	52.62	104.3
18	55.77	110.6
19	58.91	116.8
20	62.05	123.0

Utilizing equation (3.1), the experimental normal displacements, w_j, at the locations, x_j, j= 1, 2, 3, ..k, in the clamped-free direction can be set equal to the series approximation for the same deformation given by equation (3.17). In this case, the chord location, y, is held constant. Doing so, yields

$$W_x(x_j) = w_j = \sum_{i=1}^{p} A_i X_i(x_j), \qquad j = 1, 2, \ldots k \qquad (k \geq p) \qquad (3.18)$$

The only unknowns in equation (3.18) are the modal weighting coefficients, A_i.

Since we require that $k \geq p$, equation (3.18) represents a set of k non-homogeneous simultaneous equations in p unknowns, i.e., the modal weighting coefficients, A_i, i = 1, 2, ...p. Equation (3.18) can be represented in terms of matrix notation as

$$[X]\ (A) = (W) \tag{3.19}$$

where [X] - rectangular k x p matrix of the p clamped-free beam mode functions at the k spanwise locations, x_j, j=1,2,....k.

([X] has rank $r \leq p$)

(A) - column vector of the p modal weighting coefficients, A_i, i=1,2..p.

(W) - column vector of the k normal displacements, w_j, j=1,2,...k, obtained at the locations x_j using holographic interferometry.

For the case of k=p, equation (3.19) can be solved for the modal weighting coefficients, (A), using standard matrix inversion techniques. But since we are dealing with data points determined experimentally and the resulting inherent error, it is prudent to choose k>p such that equation (3.19) represents an overdetermined set of equations. A "best fit" approximation for the p weighting coefficients, (A) can then be found based on the k normal displacement values, (W). This type of approach was implemented using a linear least square solution of equation (3.18) as described in the following.

The k simultaneous linear equations given by equation (3.18) can be solved for the p modal weighting coefficients, (A), for the case of k>p by utilizing the concept of the pseudoinverse developed by E. H. Moore (Ref. 9) and R. Penrose (Ref. 10) independently of each other under the name "generalized inverse". At present, the name "pseudo-inverse" is the more generally accepted term and will be used in the following discussion. The application of the pseudoinverse to the solution of overdetermined systems of linear equations is dealt with quite lucidly by Golub and Reinsch (Ref. 11).

The rectangular (k x p) matrix, [X], appearing in equation (3.19) can be represented as (Ref. 11)*

$$[X] = [U]\,[\Sigma]\,[V]^T \tag{3.20}$$

where [U] - matrix of the p orthonormalized eigenvectors of $[X]\,[X]^T$ based on the p largest eigenvalues of $[X]\,[X]^T$

[V] - matrix of the p orthonormalized eigenvectors of $[X]^T\,[X]$

$[\Sigma]$ - Diag $(\lambda_1, \lambda_2, \ldots)$ is the diagonal matrix consisting of the non-negative square roots of the eigenvalues of $[X]^T\,[X]$.

$[\Sigma]$ is arranged such that $\lambda_1 > \lambda_2 > \ldots \lambda_p$

$[U]^T\,[U] = [V]^T\,[V] = [V]\,[V]^T = I_p$ (I_p is the identity matrix)

---

* $[V]^T$ - Transpose of matrix [V].

The pseudoinverse of [X], denoted as $[X]^+$, is obtained from the expression

$$[X]^+ = [V]\,[\Sigma]^{-1}\,[U]^T \tag{3.21}$$

where [U] and [V] are defined as before and

$$[\Sigma]^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 & 0 & . & . & . \\ 0 & \frac{1}{\lambda_2} & 0 & 0 & . & . & . \\ . & . & . & 0 & 0 & \frac{1}{\lambda_p} & \end{bmatrix}$$

$$[\Sigma]\,[\Sigma]^{-1} = [\Sigma]^{-1}\,[\Sigma] = I_p$$

equation (3.21) can be used to solve for the modal weighting coefficients in equation (3.19).

Pre-multiplying by $[X]^+$ yields:

$$[X]^+\,[X]\,(A) = [X]^+\,(W) \tag{3.22}$$

$$[I_p]\,(A) = [X]^+\,(W) \tag{3.23}$$

$$(A) = [V]\,[\Sigma]^{-1}\,[U]^T\,(W) \tag{3.24}$$

Equation (3.24) gives the p weighting coefficients based on the k input displacements and the p clamped-free beam functions evaluated at the locations x_j, j=1,2,...k. The nature of this solution is indeed fortuitous for the following two reasons (Ref. 12). First, if the rank, r, of matrix [X] is r=p, then the weighting coefficient vector (A) in equation (3.24) is the "best fit" in the sense that it is nearest to the actual vector (A) in a least squares sense. Second, if the rank of matrix [X] is r<p, then there is no exact solution

but equation (3.24) still gives the best solution in the sense of least squares, i.e., [X](A) is as close to (W) as possible. Appendix A gives a simple example of computing the pseudoinverse of a 3x2 matrix.

The computation of the psudoinverse of [X] in equation (3.19) and the resulting solution for the modal weighting coefficients, (A), was carried out using the computer program LLSQAR that is on permanent file on the computer system at Wright-Patterson AFB. The routine is part of the Computer Science Center's International Mathematical Scientific Library package. A brief description on the actual mechanics of using LLSQAR is given in Appendix D.

With the values of the modal weighting coefficients determined, equations (3.1) and (3.7) can be used to determine the displacement and curvature, respectively, for any location, x, y=constant, along the span of the plate.

In a manner exactly paralleling the preceding development for plate deformation based on a fixed-free modal series, the modal weighting coefficients appearing in equation (3.3) for a free-free beam series can be determined by using the experimental normal displacements, w_j, at each of k locations, y_j, j=1,2,3...k, in a line across the chord of the plate, in the free-free direction. In this case, the span or x location of the plate will be held constant. Carrying this out, yields the equation

$$W_y(y) = w_j = \sum_{i=1}^{q} B_i Y_i(y_j) \qquad (3.25)$$

for i=1,2,....q and j=1,2,....k $\quad (k \geq q)$

The modal weighting coefficients, B_i, are then computed using the computer algorithm LLSQAR. Once this has been accomplished, the displacement and curvature at any point y, x=constant, can be computed via equations (3.3) and (3.9). Section IV of this report presents a detailed discussion of the steps involved in applying the method to a typical plate vibration mode shape.

3.3 Ramifications of Using Simple Beam Eigenfunctions

At the beginning of this section mention was made of the fact that while the clamped-free and free-free beam functions satisfy the corresponding plate geometric boundary conditions exactly, the plate's natural boundary conditions along its free edges are only approximately satisfied. The ramifications of this can be seen by looking at the natural boundary conditions that must be satisfied along the tip of the cantiliver plate shown in Figure 3.1. Satisfying equilibrium along this edge requires that the shear forces in the x direction and the moments about the y axis be zero. Hence, we require that:

$$M_x\Big|_{x=a} = D\left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}\right)\Bigg|_{x=a} = 0 \tag{3.26}$$

$$V_x\Big|_{x=a} = D\left[\frac{\partial^3 W}{\partial x^3} + (2 - \nu) \frac{\partial^3 W}{\partial x\, \partial y^2}\right]\Bigg|_{x=a} = 0 \tag{3.27}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the plate flexural stiffness. Now the clamped-free beam function given by equation (3.2) has the property that the first term in each of equations (3.26) and (3.27) is identically equal to zero, i.e.,

$$\frac{\partial^2 W}{\partial x^2} = \frac{\partial^3 W}{\partial x^3} = 0, \qquad x = a \tag{3.28}$$

when W(x) is approximated by a series of clamped-free beam functions. This means that the calculated moment and, hence the strain, ε_x, at the plate free edge will have an error that is proportional to $y \frac{\delta^2 W}{\delta y^2}$ and the calculated shear will have an error that is proportional to $(2-\nu)\frac{\delta^3 W}{\delta x \delta y^2}$. There will be a similar error in the value of the chordwise strain, ε_y, at the plate's free edges y=o and b. Fortunately, the magnitude of this inherent error is not appreciable as will be seen in Section IV.

SECTION IV

INTERACTIVE COMPUTER PROGRAM - HOLOCURVE

The technique developed in Section III for determining the displacement and curvature of a vibrating plate-like structure was programmed for use on a CDC6600 computer via a Tektronix 4010 interactive computer terminal. By using the Tektronix computer unit, one has the advantage of interacting with the computer program while it is executing on the CDC6600 computer. In this way, input data can be deleted or modified, decisions as to program direction can be made, and, most importantly, the results are returned to the user immediately in either tabular or plot format.

Figure 4.1 shows a flow chart of the computer program called HOLOCURVE. A listing of the program is given in Appendix B. The questions that the user must answer while executing HOLOCURVE are listed below in order of appearance on the Tektronix 4010 screen along with possible answers (in brackets) and a brief discussion of each entry.

1. WHAT IS THE MODE NAME (8H)? - - -

   Input any letter/number identifier that takes up eight spaces or less.

2. WILL OUTPUT BE BASED ON CLAMPED-FREE (CF) OR FREE-FREE (FF) BEAM SERIES? TYPE CF OR FF. - - - [CF / FF]

   Program is henceforth keyed to either a clamped-free or free-free modal beam series.

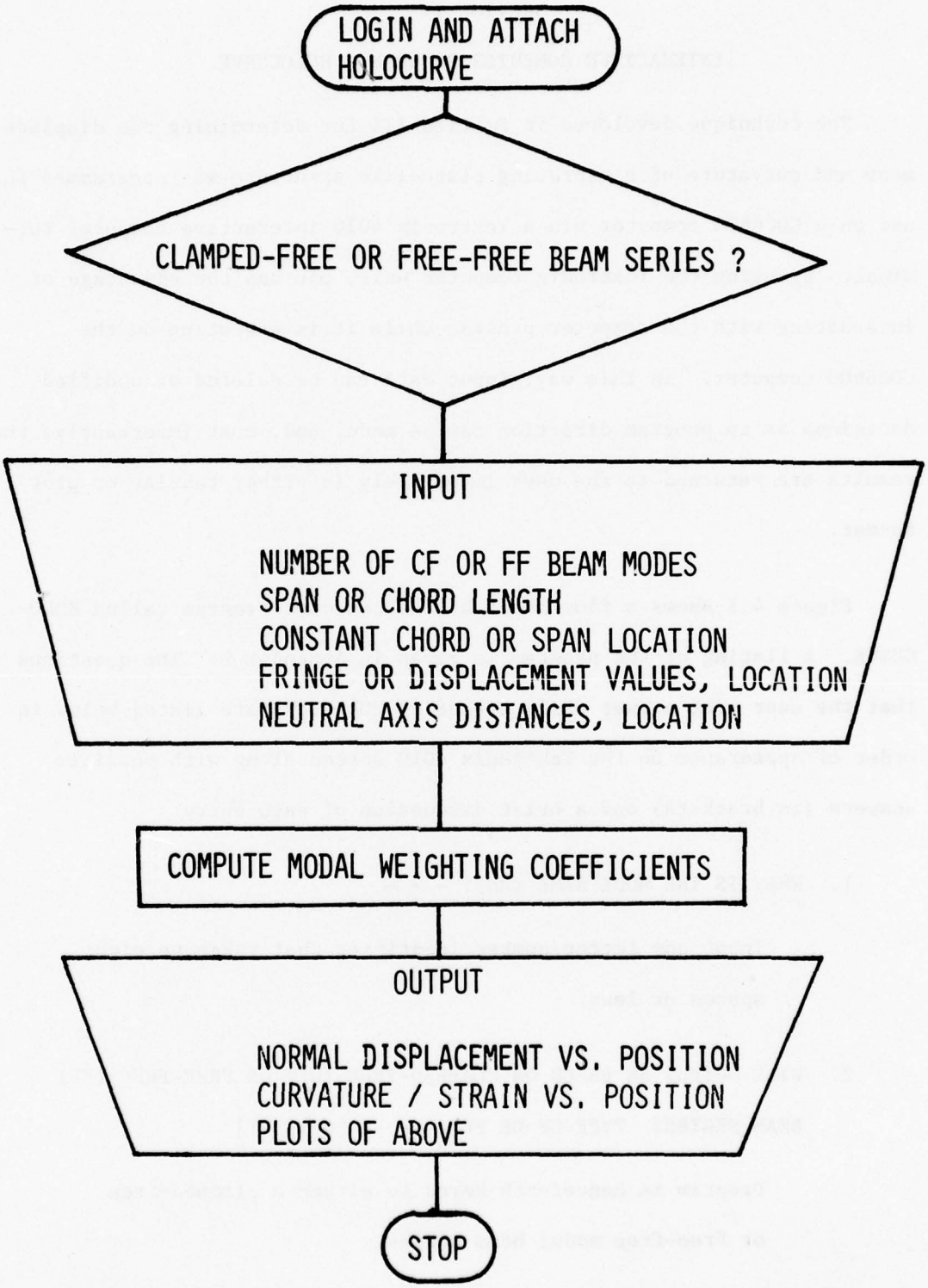


Figure 4.1 - Flow Chart for Program HOLOCURVE

3. HOW MANY CF [OR FF] BEAM SERIES TERMS WILL BE USED? - - -

[Floating point whole number]

This specifies the value of p in equation (3.1) for a clamped-free beam or of q in equation (3.3) for a free-free beam. For the lower mode shapes (first and second bending, first and second torsion, lyre or two-stripe), a value of p = 5 or Q = 5 gives adequate results.

4. INPUT NUMBER OF FRINGE VALUES. - - -

Specifies number of discrete holographic fringe value data points that will be used.

5. INPUT SPAN [OR CHORD] LENGTH. - - - [Floating point number]

For the clamped-free case, the actual length of the span segment is input. This corresponds to the beam length, a, in equation (3.2). For example, the distance from the root to the tip of a compressor blade could be entered. For the free-free case, the length of the chord segment is input. This corresponds to the beam length, b, in equations (3.5) and (3.6). It could, for example, be the distance from the leading edge to the trailing edge of a compressor blade.

6. INPUT CONSTANT CHORD (Y) [OR SPAN (X)] LOCATION. - - -

This establishes the constant y or x location on the plate or blade for the case of a clamped-free or free-free beam series approximation, respectively. (See Figure 3.1)

7. INPUT PAIRS OF SPAN [CHORD] LOCATION, FRINGE VALUE DATA.
FREE FIELD FORMAT WITH EACH VALUE SEPARATED BY COMMAS.

DATA = [1, -5., 2, -11., 3., -12., 4.1, -8, 4.94, 0., 6., 9.,
7., 16.]

Pairs of span (or chord) location vs. holographic fringe value are input. Seven such pairs of data are input in the example shown. Note that the fringe values are positive or negative whole numbers. Note, also, there is a nodal point at x = 4.94 where the fringe value is given as zero. The HOLOCURVE subroutine, CONVRT, converts the fringe values to normal displacements using the expression given in equation (3.17).

8. WILL DISTANCE FROM NEUTRAL AXIS VS. SPAN [OR CHORD] LOCATION BE INPUT - YES OR NO? - - -

If "YES", the following two statements must be answered:

HOW MANY (SEGMENT LOCATION/NEUTRAL AXIS DISTANCE) VALUES WILL BE INPUT (MAX. OF 20)? - - - [5.]

INPUT 5. PAIRS OF (SEGMENT LOCATION, NEUTRAL AXIS DISTANCE) VALUES.

DATA = [0., .25, 1., .28, 3., .30, 5., .25, 6., .20, 7., .15]

In order to compute the bending strain at a point on the plate or blade, the distance from the neutral axis must be known (see, for example, equations (3.12) and (3.13)). In the case

of a flat plate-like structure having constant thickness, this distance is merely half of the thickness. In the case of a twisted or curved plate-like body having variable thickness, however, the neutral axis distance is not so easily computed and is entered in discrete form as in the above "DATA" entry. When the bending strains are computed along the span or chord section being analyzed, a subroutine (INTERP) utilizes a second degree polynomial interpolation scheme to compute the neutral axis distance based on the discrete input data. Appendix E. contains a more complete description of subroutine INTERP.

9. DO YOU WISH TO SEE INPUT DATA (YES OR NO)? - - -

   A listing of fringe values and their location, and, if used, neutral axis distance values and their location are displayed.

10. COMPUTATION COMPLETED FOR [Mode Name] MODE, DATA SET = 1. DO YOU WISH TO SEE MODAL WEIGHTING COEFFICIENTS FOR THE CF[OR FF] BEAM SERIES - YES OR NO? - - -

    An affirmative answer yields a listing of the modal weighting coefficients given in equations (3.1) or (3.3).

11. DISPLACEMENT AND CURVATURE AS A FUNCTION OF POSITION WILL NOW BE LISTED AND PLOTTED FOR MODE [Mode Name].

    HOW MANY DATA POINTS DO YOU WANT (MAX. OF 40)? - - -

    Chord or span segment is evenly divided into number of data points entered. Program then proceeds to give a listing of

the normal displacement as a function of position and a plot of same. It then gives a listing of the curvature (bending strain) as a function of position, and an accompanying plot.

12. IS THERE MORE DATA TO BE ANALYZED (YES OR NO?) - - -

If answer is affirmative, cycle of questions is repeated; if negative, HOLOCURVE completes its execution and user may "LOG OFF" from terminal.

Appendix C. gives a typical example of the above statements being used during a HOLOCURVE execution.

SECTION V

DISPLACEMENT AND STRAIN DETERMINATION USING HOLOCURVE

5.1 Displacement and Strain Distribution of a Vibrating Plate

In order to check the effectiveness and accuracy of the holographic data reduction technique utilized by HOLOCURVE, three natural modes of vibration of a cantilevered flat plate were determined using holographic interferometry and the resulting fringe data was input to HOLOCURVE to determine the displacement and bending strain at selected locations for each of the three vibration modes. The results from this analysis were compared to those of an eigenvalue analysis carried out on a finite element model of the plate using the multi-purpose finite element computer program, NASTRAN (Rigid Format 3).

The plate was made of 6061-T6 Aluminum and measured 3" by 7" with a thickness of .19". It was fixed along one of its short sides by clamping it between two massive steel blocks. Time average holograms were made of each of the three natural plate modes of vibration while the plate was being excited by a 40-watt siren. A schematic of the holographic set-up is shown in Figure 5.1. The set-up is quite typical of the type used for doing holographic interferometry. A 50mw continuous wave HeNe laser (Spectra-Physics 125A) was used for the tests with the holograms being recorded on 4"x5" photographic plates (Agfa-Gevaert 10E75).

What follows are the results of using HOLOCURVE to get the displacement and corresponding bending strain for the plate vibration modes--second bending, second torsion, and the lyre mode.

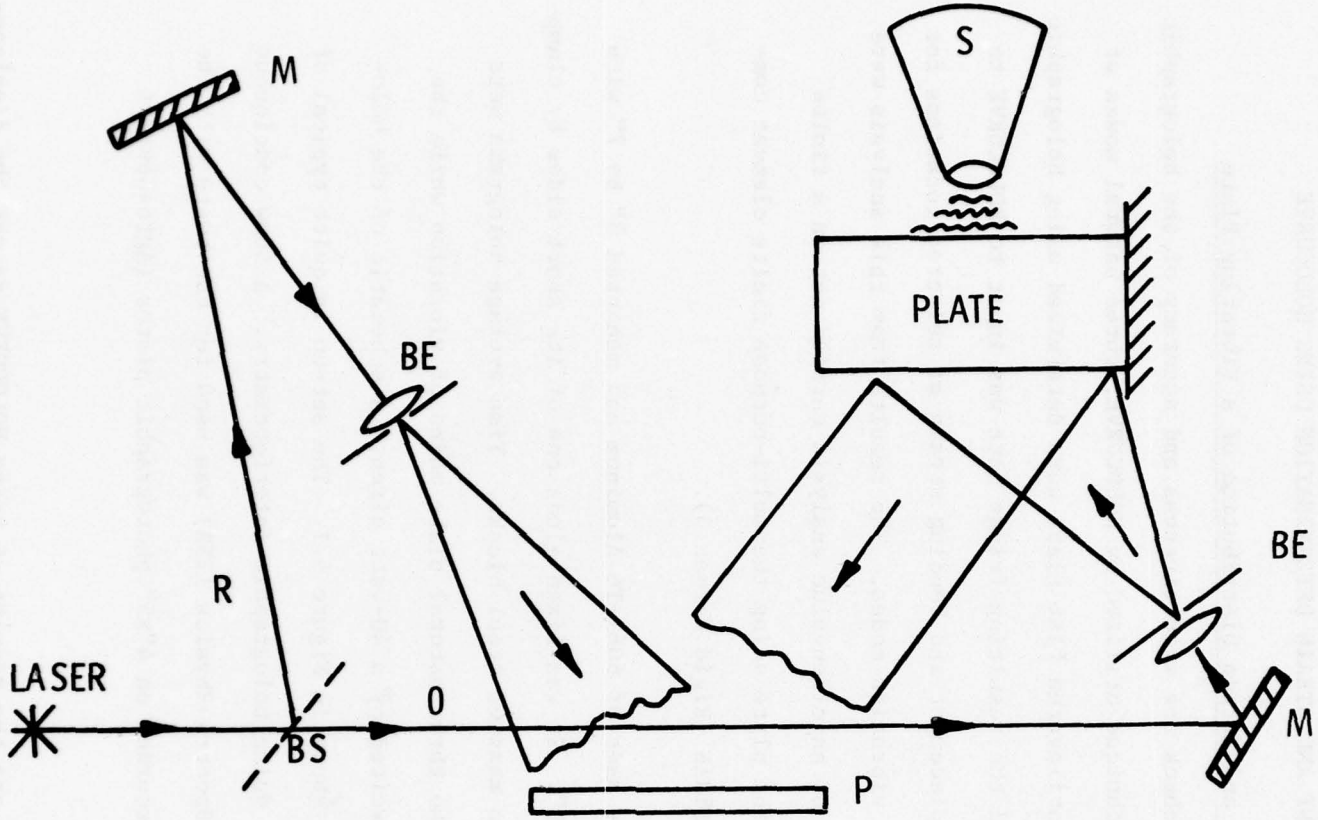


Figure 5.1 - Schematic of Experimental Set-up; BS-beam splitter, M-mirror, BE-beam expander, P-photographic plate, S-siren, R-reference beam, O-object beam

5.2 Lyre Mode of Vibration

Figure 5.2 shows the lyre mode (after the Greek instrument) of the cantilever plate obtained using holographic interferometry and finite element analysis. Note that the natural frequencies determined by each of the techniques differ by only 1.2%. In the figure, the plate is clamped along its bottom edge. The lyre mode, or a form similar to it, is a common one in turbine engine compressor blading.

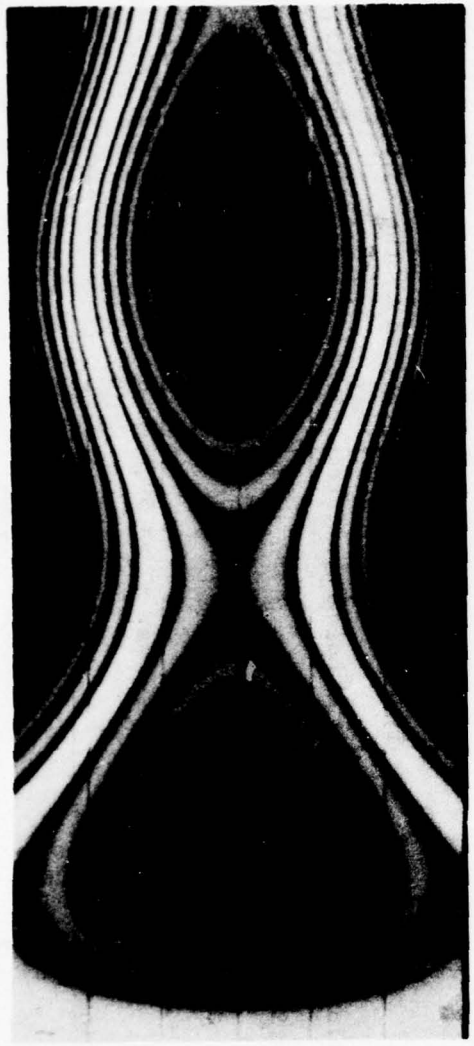
Referring to Figure 5.2, let a Cartesian coordinate system originate at the lower right-hand corner of the plate with the x axis lying along the plate's right edge and the y axis lying along the bottom (fixed) edge of the plate. Using this coordinate system, the displacement and corresponding bending strain were determined with respect to the x coordinate (spanwise direction) while keeping y (the chordwise direction) constant at y=.5". Note that the coordinate system just defined is the same as the one shown in Figure 3.1 of Section III, Theoretical Development.

Referring to the lyre mode shape characterized by the holographic fringes in Figure 5.2, the holographic fringe values as a function of x, y=.5", were determined and are given in Table 5.1.

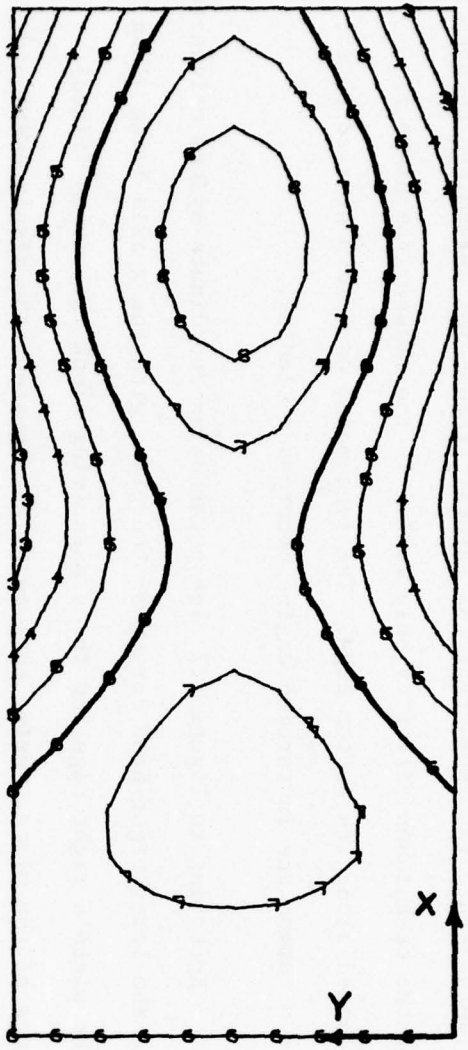
TABLE 5.1

Fringe Values vs. x, y=.5", for Lyre Mode

Fringe Number, N	Span Location, x, (in.)	Fringe Value, F
1	.50	1
2	1.25	2
3	2.50	-2
4	3.50	-4
5	4.30	-2
6	5.32	0
7	6.25	-2
8	6.95	-5



Experimental, f = 4583 Hz



Finite Element, f = 4529 Hz

Figure 5.2 - Lyre Mode of Vibration for Cantilever Plate

The fringe values and span locations given in Table 5.1 are not the complete set of values occurring on this spanwise segment of the plate and, hence, Table 5.1 does not represent a unique set of fringe value data for the span. Some general rules of thumb were used, however, in their choice.

1. Fringe value locations should be chosen such that they are approximately evenly spaced over the segment length.

2. The total number of fringe values used must be greater than the number of beam series functions input to HOLOCURVE. This will insure that full advantage is taken of the "smoothing" characteristics of the linear least squares subroutine, LLSQAR.

3. When possible, specify the locations of the nodal lines with the exception of the clamped end nodal line in the case of the clamped-free beam series approximation. To do so in this latter instance would be redundant since zero displacement is one of the geometric boundary conditions of the clamped-free beam functions.

4. Finally, regarding the number of beam functions specified for HOLOCURVE, in the case of clamped-free beam series approximation, the number of beam series functions used should be greater than or equal to the number of node lines crossing the span plus two, i.e.,

   Number CF Series Functions = Number Node Lines + 2.

   For the case of a free-free beam series approximation, the number of beam series functions used should be greater than or equal to the number of node lines crossing the chord plus three. i.e.,

   Number FF Series Functions = Number Node Lines + 3.

The set of eight fringe values and their span locations given in Table 5.1 were input to HOLOCURVE where they were used to compute the modal weighting coefficients of a five-term clamped-free beam series. The computed normal displacement is shown as the smooth curve in Figure 5.3 as a function of the span, x. Also shown in the figure are the normal displacement values computed from the finite element model of the plate using NASTRAN. The finite element values were normalized to the maximum displacement at the

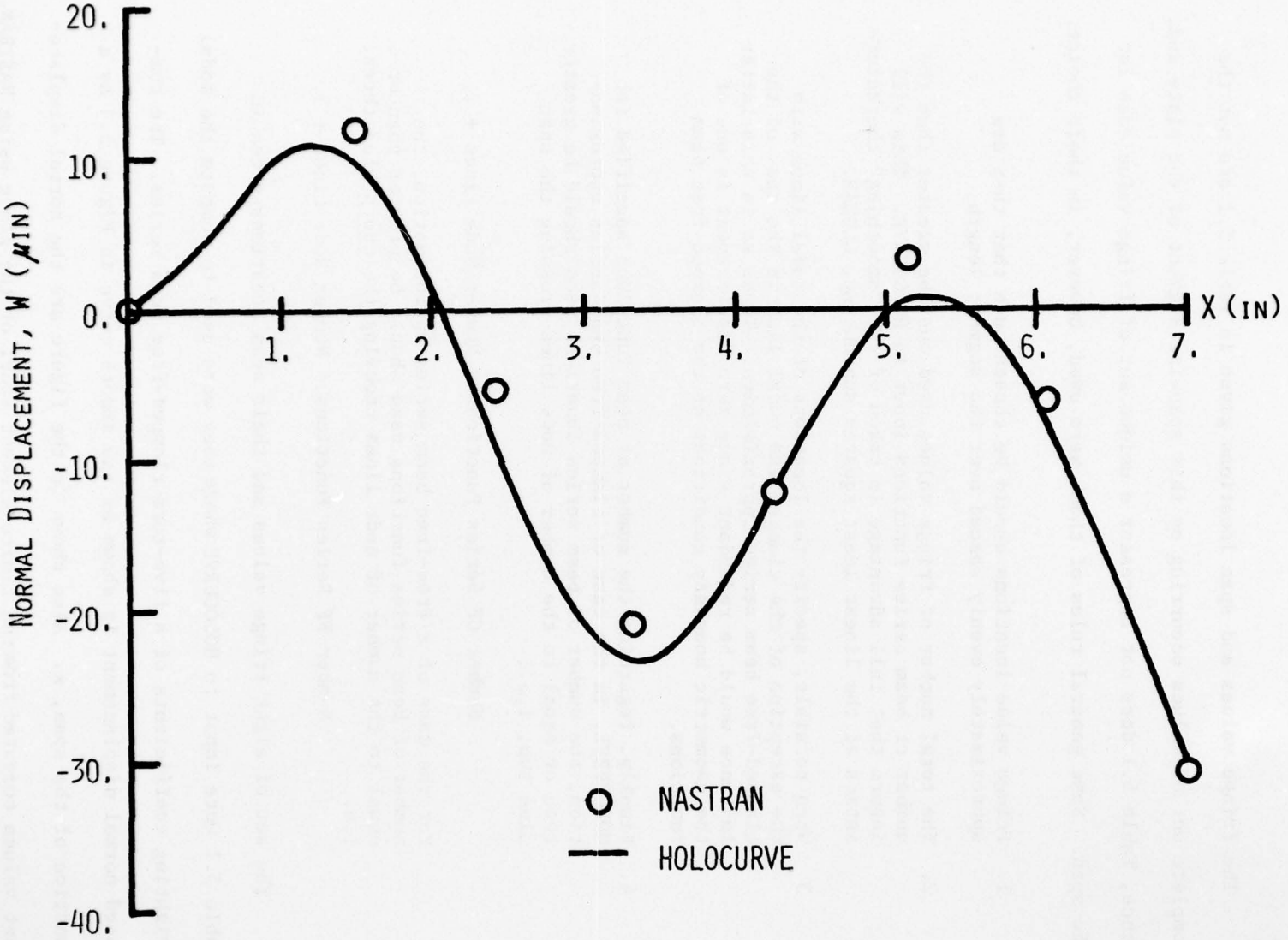


Figure 5.3 - Normal Displacement, W, vs. Span, X, at Y=.5" for Lyre Mode

plate tip as determined by HOLOCURVE. It is seen that the agreement between the two methods is quite good. The modal weighting coefficients upon which the HOLOCURVE results are based are listed in Table 5.2.

TABLE 5.2

Modal Weighting Coefficient for CF Series Approximation of Lyre Mode

CF Beam Mode, i	Modal Weighting Coefficient, $A_i \times 10^6$
1	-8.66
2	-2.860
3	-1.886
4	9.164
5	1.526

Note that coefficients A_1, and A_4, representing the first and fourth clamped-free beam bending mode shapes, respectively, are the dominant terms in the series. This is as one would expect, since the first order of deflection of the mode is that of a negative first bending mode while the actual modal displacement is that of a positive fourth bending mode shape, i.e., four nodal lines.

Figure 5.4 shows the resulting bending strain, ε_x, in the plate's x direction computed from the second derivative of the five-term clamped-free beam series. As in the plot of the normal displacement, the individual data points in the figure represent results from the NASTRAN finite element analysis of the plate.

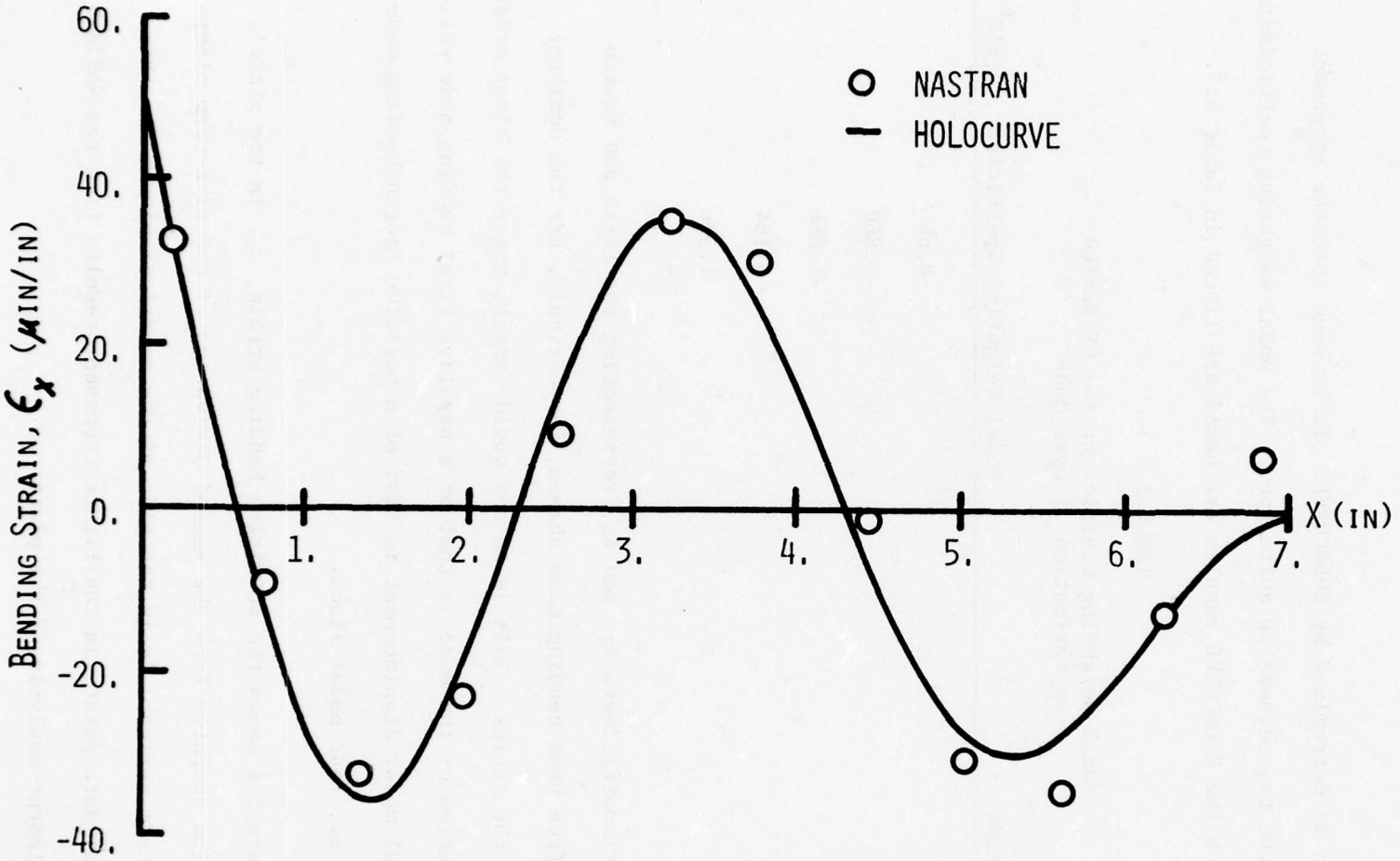


Figure 5.4 - Bending Strain, ϵ_x, vs. Span, X, at Y=.5" for Lyre Mode

The values of the bending strain, computed using NASTRAN, were normalized to the HOLOCURVE output at the span location, x=.15". Again, the agreement is seen to be quite good.

The displacement and resulting bending strain were also determined for the lyre mode in Figure 5.2 as a function of the width, y, for x=2.5". The holographic fringes upon which the HOLOCURVE computations were based are given in Table 5.3.

TABLE 5.3

Fringe Values vs. y, x=2.5", for Lyre Mode

Fringe Number, N	Chord Location, y (in.)	Fringe Value, F
1	.06	-5
2	.48	-2
3	.80	0
4	1.20	2
5	1.84	2
6	2.25	0
7	2.53	-2
8	2.96	-5

This set of eight fringe values and chord location data were used in HOLOCURVE to compute the modal weighting coefficients of a 5-term free-free beam series. The normal displacement curve computed from this series is shown as the smooth curve in Figure 5.5 along with the normalized grid point displacements from the NASTRAN finite element calculations. Note that the finite element computations yield a greater value for the displacement at the chord midpoint. This maximum

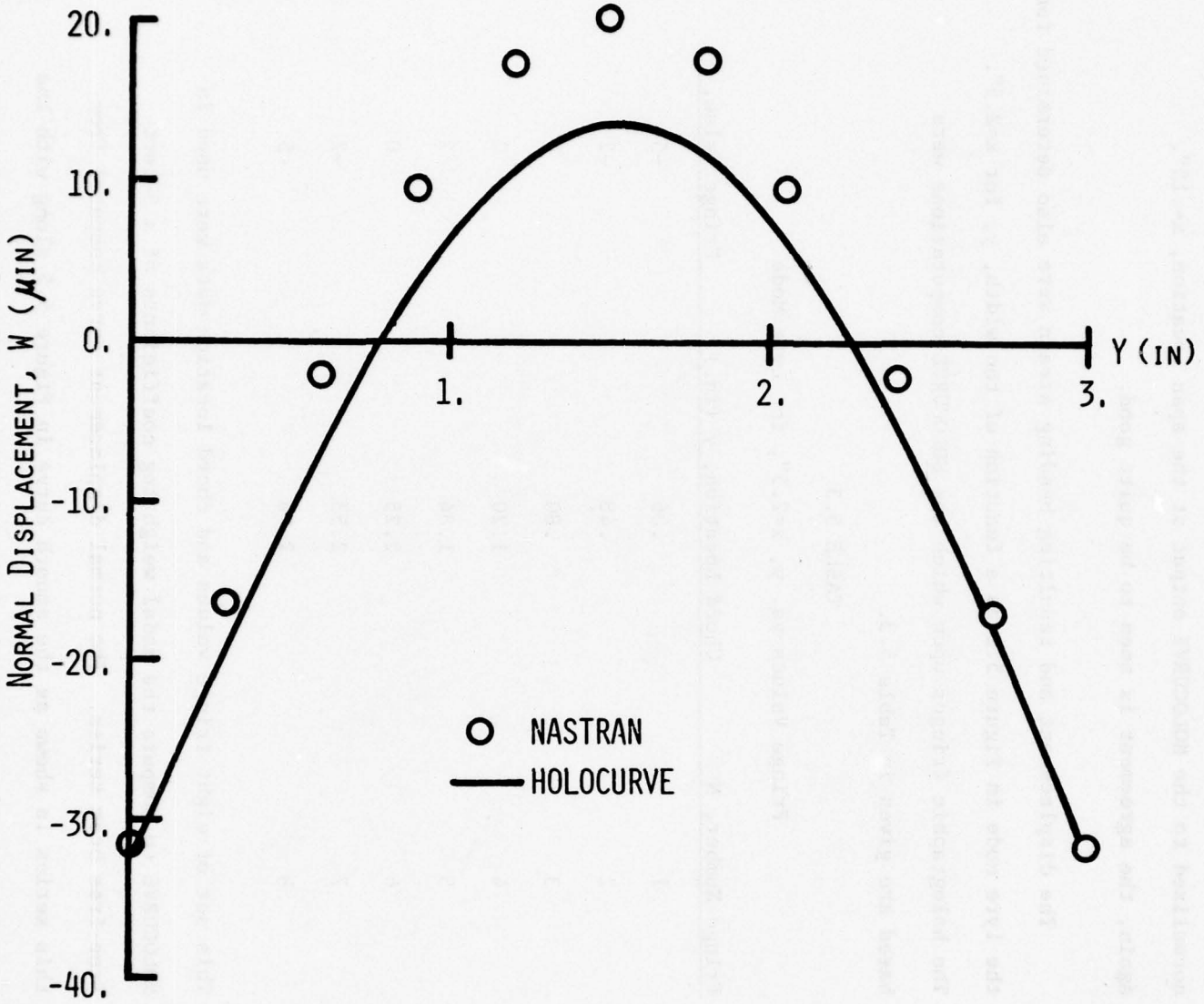


Figure 5.5 - Normal Displacement, W, vs. Chord, Y, at X=2.5" for Lyre Mode

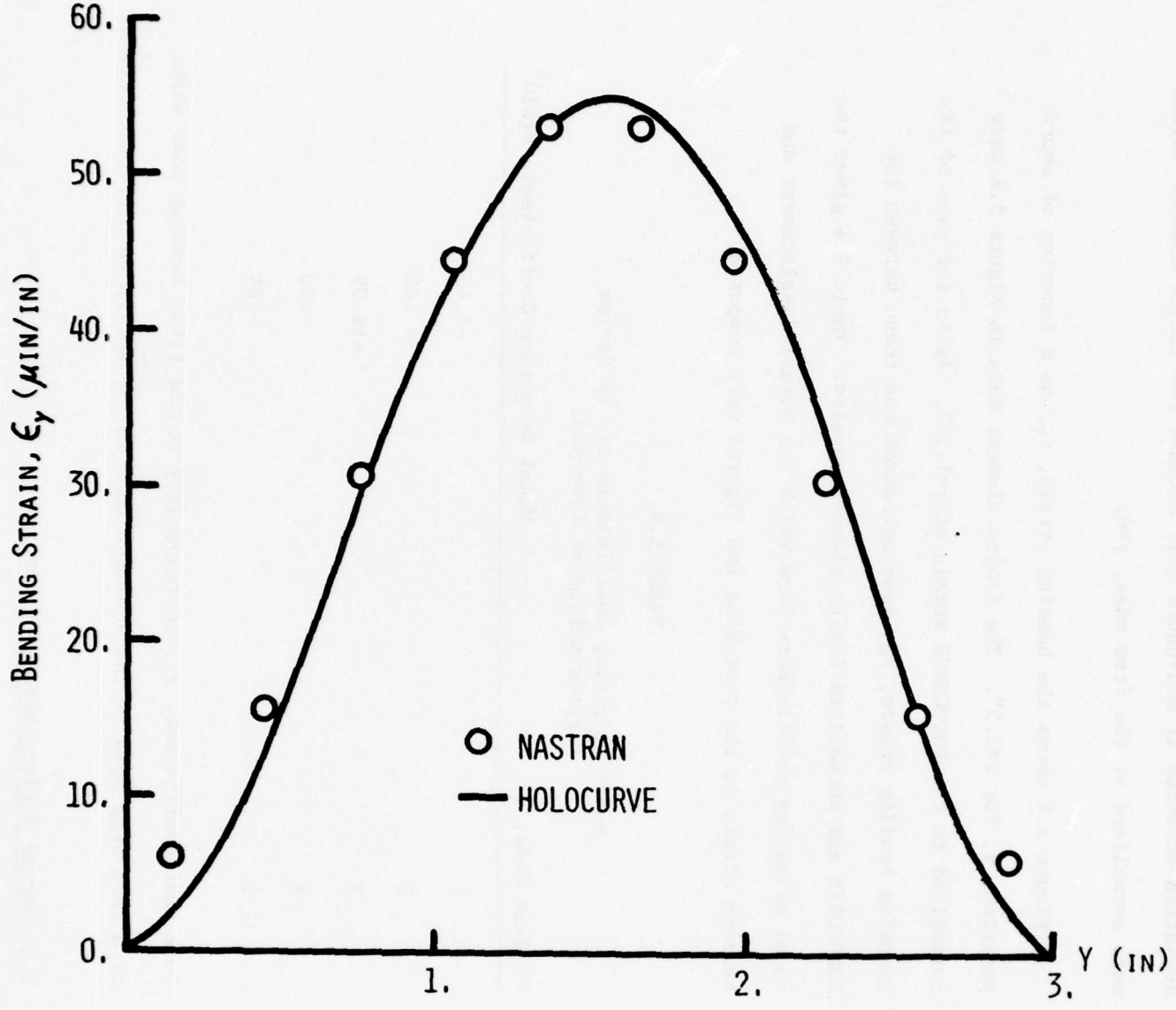


Figure 5.6 - Bending Strain, ϵ_y , vs. Chord, Y, at X=2.5" for Lyre Mode

deviation occurred at mid-chord only because the finite element values were normalized at the free edge, y=0.

Figure 5.6 shows the bending strain, ε_y, as a function of chord position, y, for x=2.5". The finite element data in Figure 5.6 were normalized to the HOLOCURVE strain at y=1.35". As in the case of the spanwise bending strain, ε_x, good agreement was found between the HOLOCURVE and normalized finite element results. Table 5.4 gives the modal weighting coefficients from which the normal displacement and bending strain in the preceding two figures were computed.

TABLE 5.4

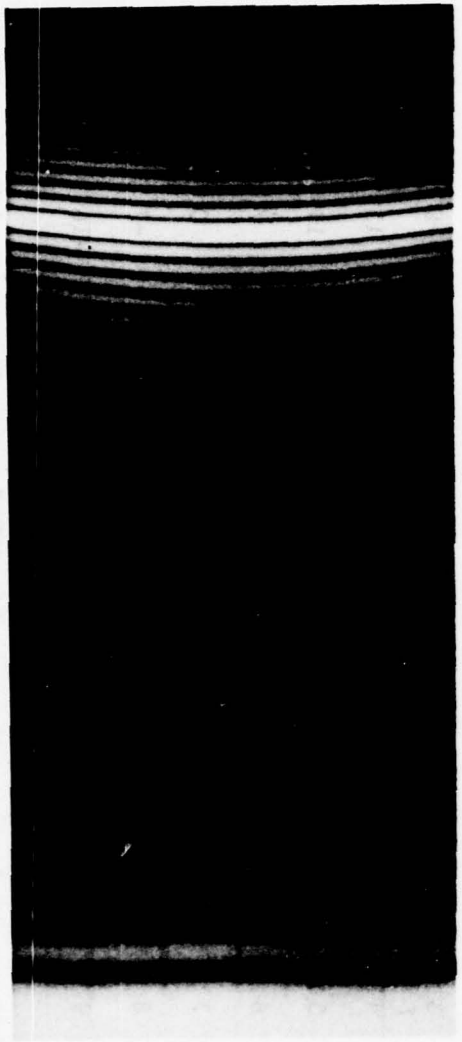
Modal Weighting Coefficients for FF Series Approximation of Lyre Mode

FF Beam Mode, i	Modal Weighting Coefficient, $B_i \times 10^6$
1	-3.45
2	- .40
3	-14.09
4	.20
5	-.04

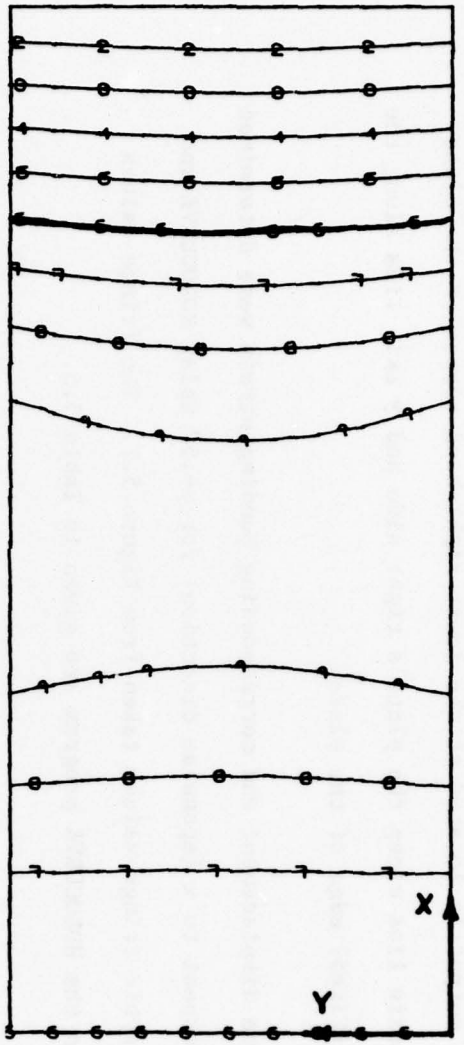
Note that coefficient, B_3, corresponding to the first bending mode shape of a free-free beam is the dominant term in the series as one would expect.

5.3 Second Bending Mode of Vibration

Figure 5.7 shows the second bending mode of vibration for the cantilever plate obtained using holographic interferometry and finite element



Experimental, f = 774 Hz



Finite Element, f = 782 Hz

Figure 5.7 - Second Bending Mode of Vibration for Cantilever Plate

analysis. As in the preceding section on the lyre mode, the coordinate system for locating fringe values and interpreting displacement and strain output is located at the lower right-hand corner of the plate. The x axis lies along the plate's right side and y axis lies along the lower (fixed) edge of the plate.

The displacement and corresponding bending strain were determined with respect to x (spanwise direction) for y=.5" using HOLOCURVE and holographic fringe values taken from Figure 5.7. The fringe values used in the HOLOCURVE program are given in Table 5.5.

TABLE 5.5

Fringe Values vs. x, y=.5", for Second Bending Mode

Fringe Number, N	Span Location, x (in.)	Fringe Value, F
1	.44	1
2	1.16	4
3	2.11	9
4	2.66	11
5	3.88	11
6	4.66	7
7	5.44	0
8	6.11	-7
9	6.98	-16

This set of data was input to HOLOCURVE and a five-term clamped-free beam series was generated. The modal weighting coefficients for this beam series are given in Table 5.6.

TABLE 5.6

Modal Weighting Coefficients for CF Series
Approximation of Second Bending Mode

CF Beam Mode, i	Modal Weighting Coefficient, $A_i \times 10^6$
1	-.68
2	50.64
3	.83
4	-.24
5	.51

Looking at Table 5.6, it is evident that the plate's second bending mode shape is quite similar to that of a simple clamped-free beam and, hence, the modal weighting coefficient, A_2, dominates the clamped-free beam series.

Figure 5.8 shows the plate's second bending mode normal displacement computed by HOLOCURVE as a function of the span coordinate, x, for y=.5". The normal displacement values determined by finite element analysis were normalized to the HOLOCURVE displacement at the plate's free edge (x=7.0"). The agreement between the two methods of analysis for this case are remarkably good.

The bending strain, ε_x, resulting from the plate displacement shown in Figure 5.8 is shown in Figure 5.9. As in the case of the lyre mode, the bending strain values computed using the finite element method were normalized to the HOLOCURVE strain value at x=.15". Again, the agreement between the two methods is seen to be quite good.

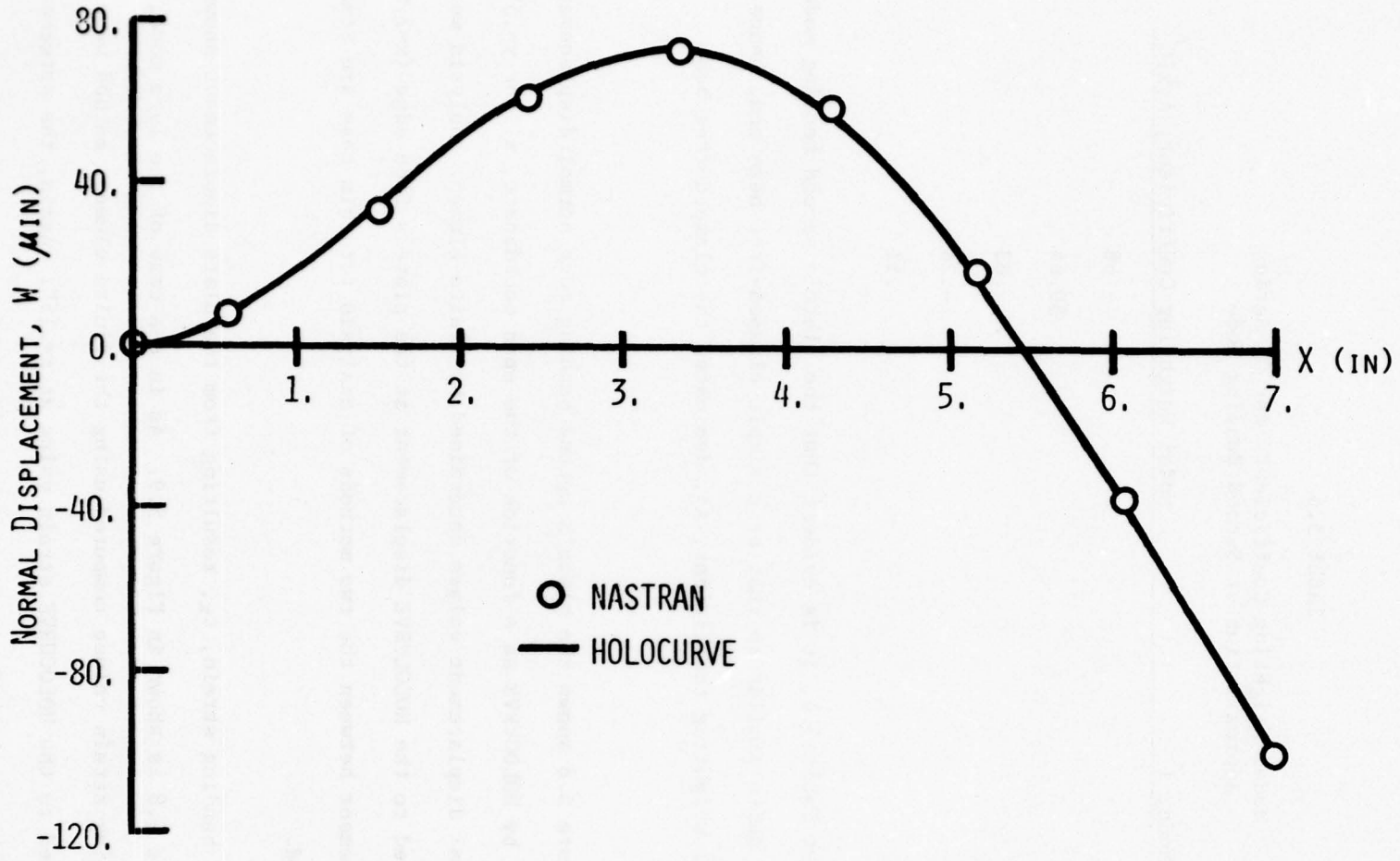


Figure 5.8 - Normal Displacement, W, vs. Span, X, at Y=.5" for Second Bending Mode

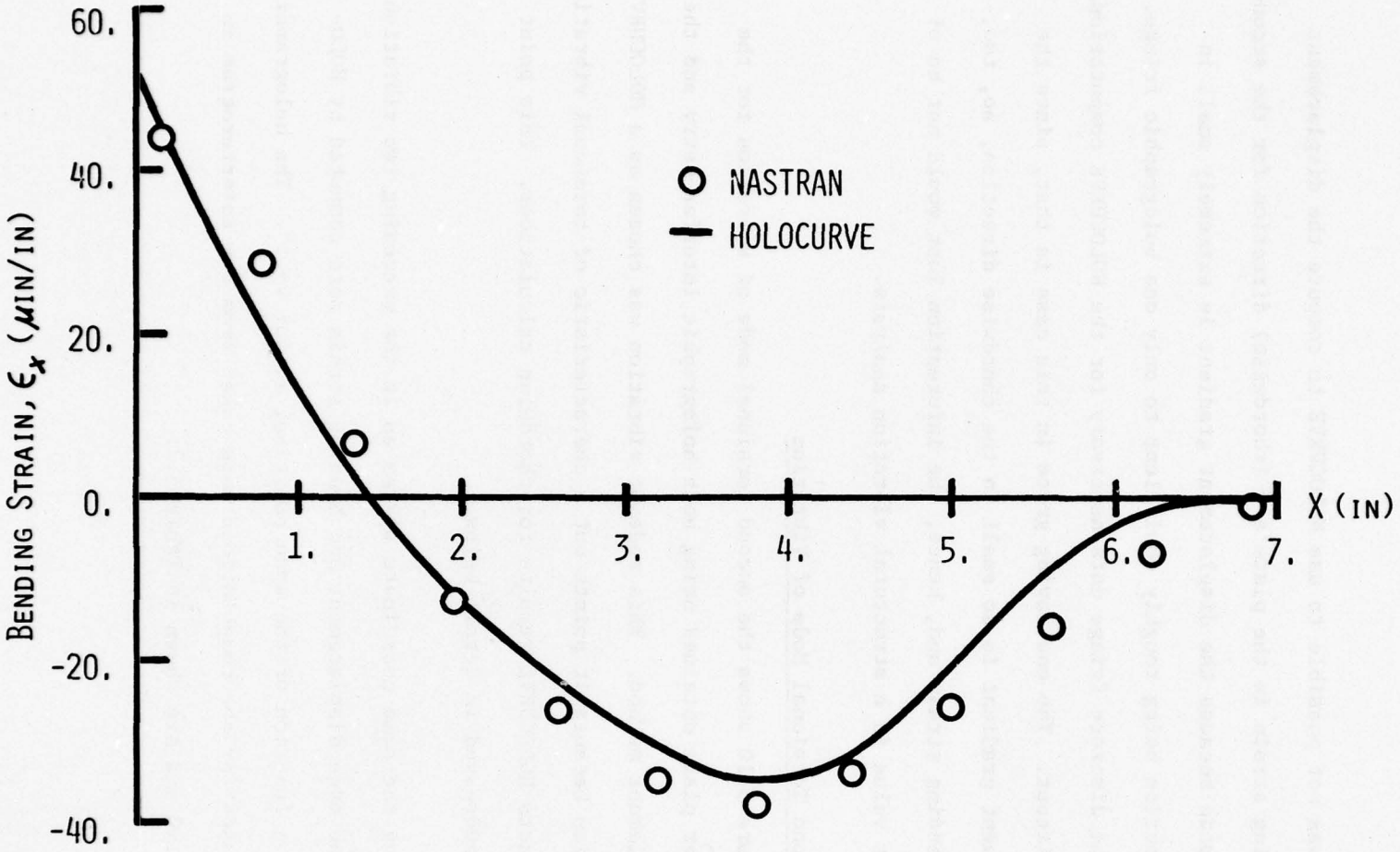


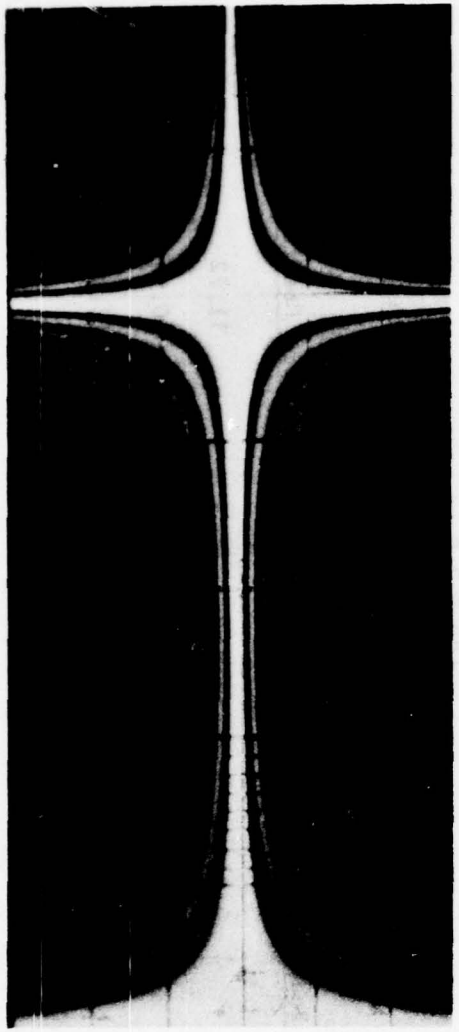
Figure 5.9 - Bending Strain, ϵ_x, vs. Span, X, at Y=.5" for Second Bending Mode

It was not possible to use HOLOCURVE to compute the displacement and bending strain in the plate's y (chordwise) direction for the second bending mode because the displacement gradient is extremely small in this direction being roughly equivalent to only one holographic fringe. Hence, the discrete fringe data necessary for the HOLOCURVE computations is nonexistent. The one saving grace in this case is that, since the displacement gradient is so small in the chordwise direction, so, too, is the bending strain and, hence, the information lost would not be of any great value in a structural vibration analysis.

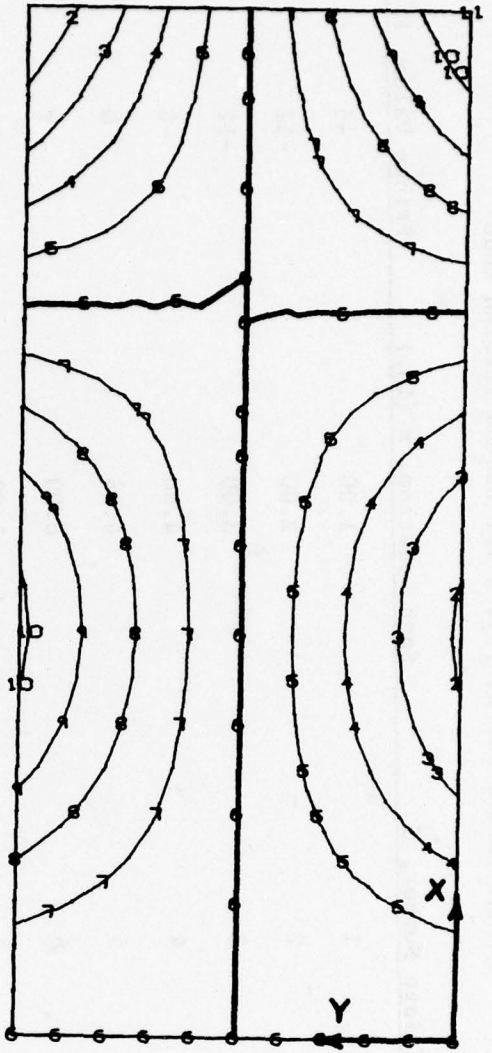
5.4 Second Torsional Mode of Vibration

Figure 5.10 shows the second torsional mode of vibration for the cantilever plate obtained using both holographic interferometry and the finite element method. This mode of vibration was chosen as a HOLOCURVE application because it points out a characteristic of torsional vibration that impacts HOLOCURVE results for chordwise calculations. This point will be addressed in detail below.

Using the same coordinate system as in the preceding two vibration modes, the mode displacement and bending strain were computed by HOLOCURVE as a function of the span position, x, for y=.5". The holographic fringes used for the computations were taken from the interferogram in Figure 5.10 and are shown in Table 5.7.



Experimental, f = 1897 Hz



Finite Element, f = 1956 Hz

Figure 5.10 - Second Torsion Mode of Vibration for Cantilever Plate

TABLE 5.7

Fringe Values vs. x, y=.5", for Second Torsion Mode

Fringe Number, N	Span Location, x (in.)	Fringe Value, F
1	1.00	-5
2	2.00	-11
3	3.00	-12
4	4.00	-8
5	4.94	0
6	6.00	9
7	7.00	16

Entering this set of data into the HOLOCURVE, the modal weighting coefficients were computed for a five-term clamped-free beam series and are shown in Table 5.8.

TABLE 5.8

Modal Weighting Coefficients for CF Series Approximation of Second Torsion Mode

CF Beam Mode, i	Modal Weighting Coefficient, $A_i \times 10^6$
1	11.72
2	-50.84
3	-12.54
4	-.46
5	-1.49

This five-term series was used to compute the normal displacement, W,

and corresponding bending strain, ε_x, which are shown as the smooth curves in Figures 5.11 and 5.12, respectively. The HOLOCURVE and finite element results are again in good agreement. As in the preceding two cases, the finite element data was normalized to the HOLOCURVE displacement with respect to the plate's free end and to the HOLOCURVE bending strain at x=.15".

An analysis was also carried out using HOLOCURVE to determine the displacement and bending strain in the plate's y (chordwise) direction with x=1.0". Table 5.9 lists the holographic fringe data used in the analysis.

TABLE 5.9

Fringe Values vs. y, x=1.0", for Second Torsion Mode

Fringe Number, N	Chord Location, Y (in.)	Fringe Value, F
1	.08	-7
2	.30	-6
3	.50	-5
4	.70	-4
5	.90	-3
6	1.10	-2
7	1.32	-1
8	1.47	0
9	1.62	1
10	1.83	2
11	2.02	3
12	2.22	4
13	2.43	4
14	2.62	6
15	2.83	7
16	3.00	8

This set of fringe data differs from the others used up to this point in that <u>all</u> of the fringes and their corresponding chord locations at x=1.0" were used.

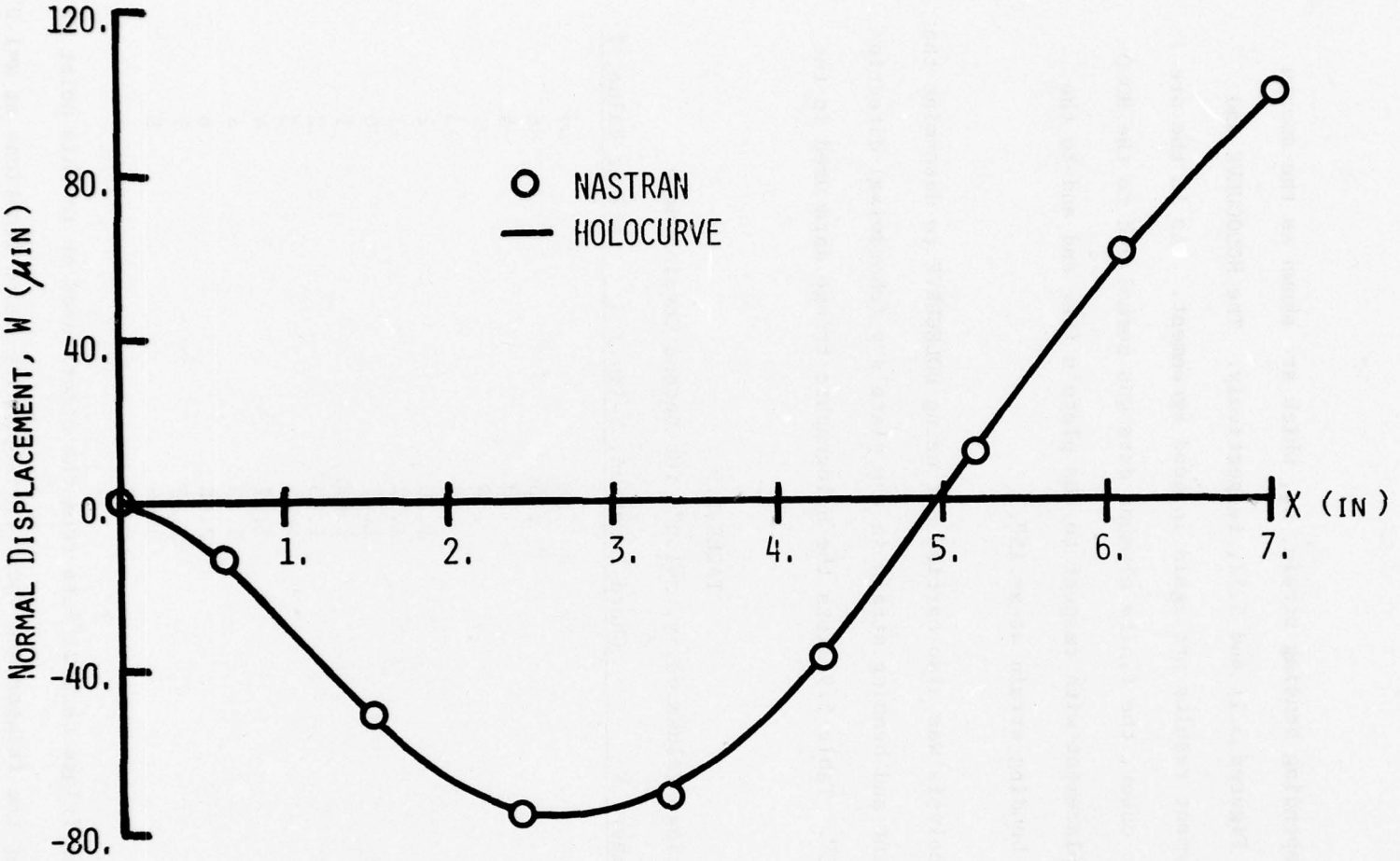


Figure 5.11 - Normal Displacement, W, vs. Span, X, at Y=.5" for Second Torsion Mode

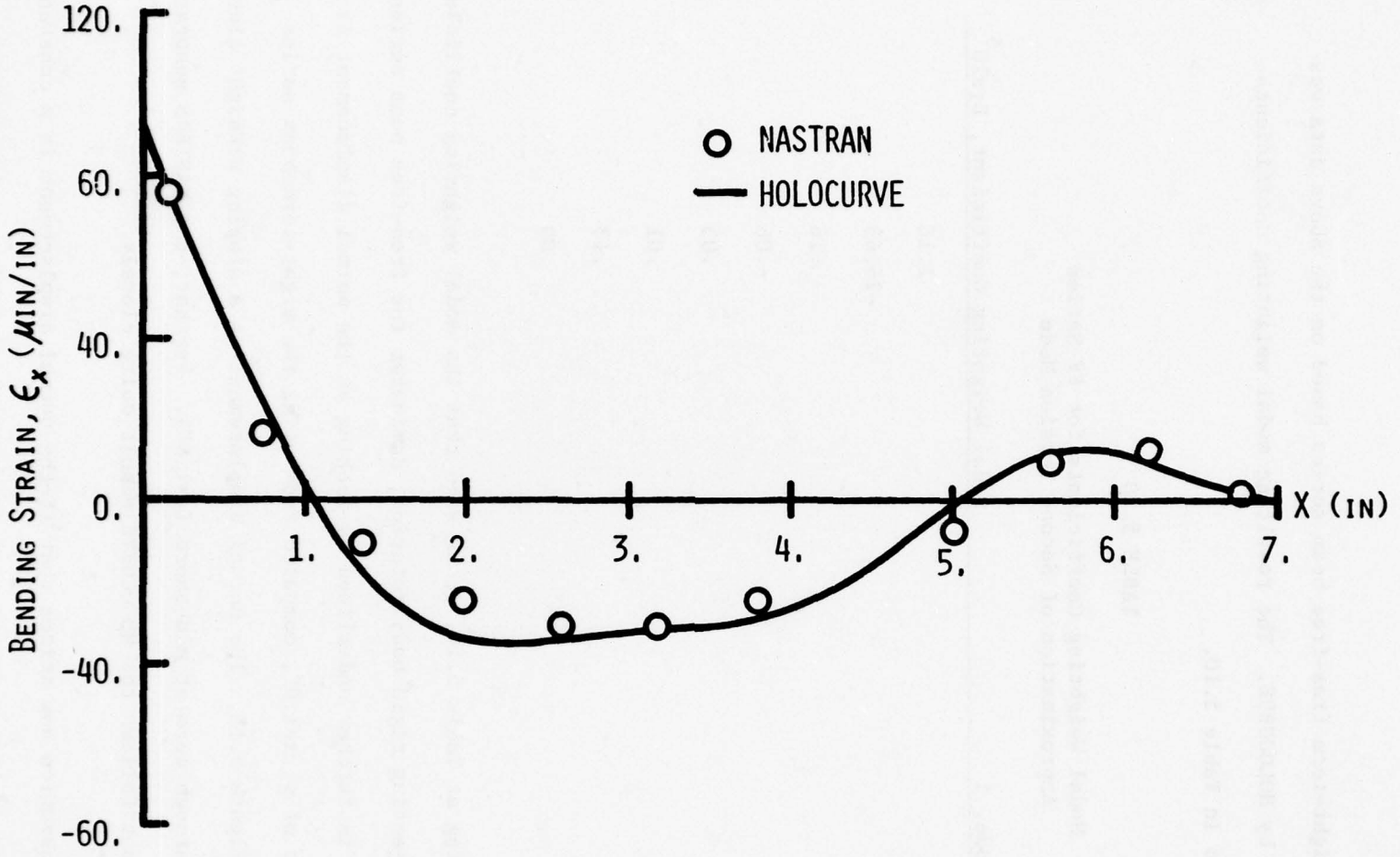


Figure 5.12 - Bending Strain, ϵ_x, vs. Span, X, at Y=.5" for Second Torsion Mode

An eight-term free-free beam series based on the above data was generated by HOLOCURVE. The resulting modal weighting coefficients are listed in Table 5.10.

TABLE 5.10

Modal Weighting Coefficients for FF Series Approximation of Second Torsion Mode

FF Beam Mode, i	Modal Weighting Coefficient, $B_i \times 10^6$
1	1.18
2	-26.63
3	.16
4	-.06
5	.03
6	.01
7	.13
8	.00

Looking at Table 5.10, it is seen that the modal weighting coefficient, B_2, representing rigid body rotation, dominates the free-free beam series. This fact is further underlined by looking at the normal displacement as a function of y, x=1.0", computed from using the eight-term beam series shown in Figure 5.13. The normal displacement is a sloping straight line passing through zero at mid-chord (y=1.5"). Further, the NASTRAN generated displacement follows the HOLOCURVE result quite closely.

The question now arises that if the normal displacement is a constant or, at most, a linear function of the spatial variable, y, (see equations 3.4 and 3.5), what meaning does it have to take the second derivative of

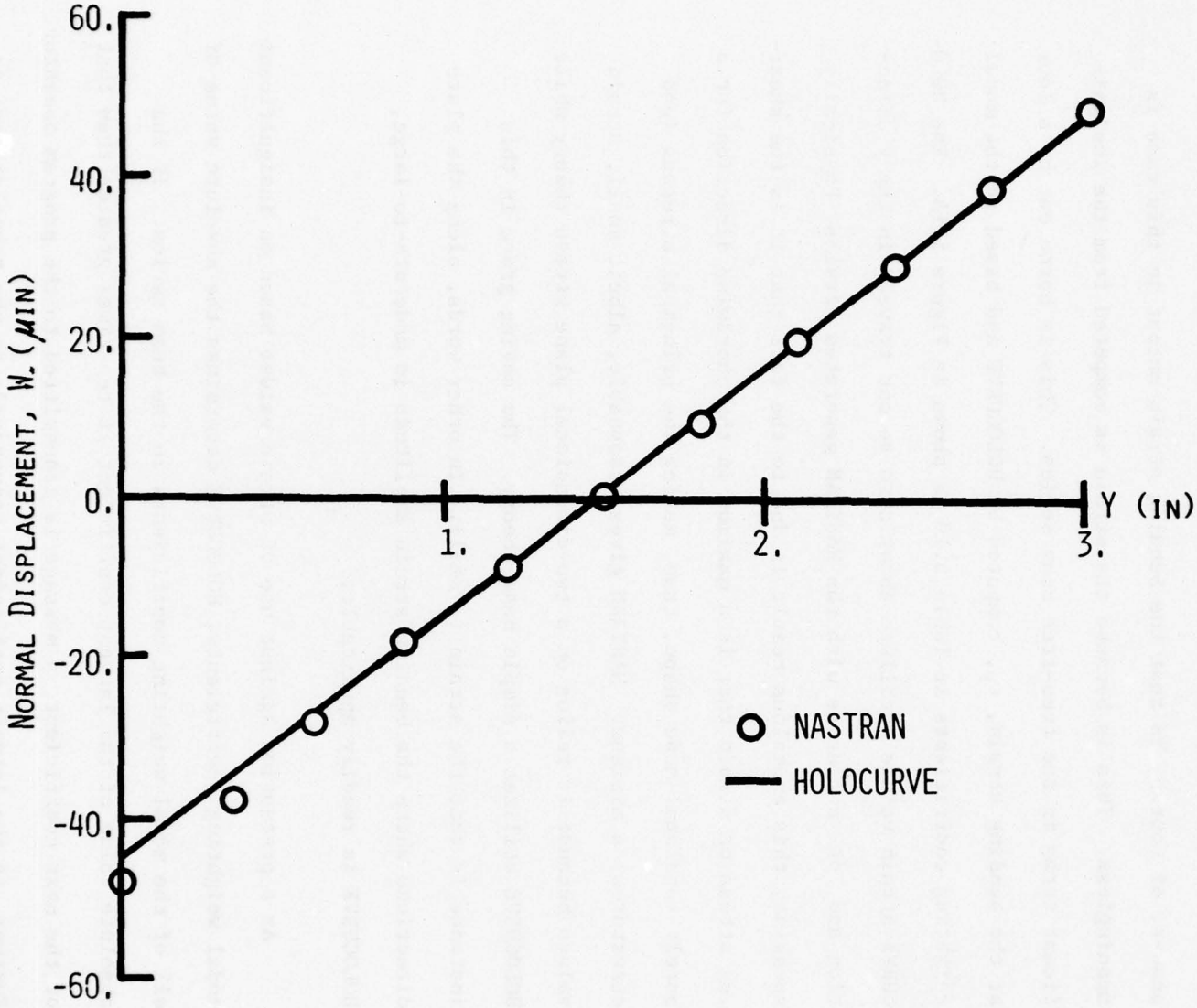


Figure 5.13 - Normal Displacement, W, vs. Chord, Y, at X=1.0", for Second Torsion Mode

the free-free beam series in order to obtain the bending strain? The answer, of course, is that the bending strain output in this case is meaningless. This is because the strain is computed from the insignificant terms in the free-free beam series. This is borne out by a look at the bending strain, ε_y, computed by HOLOCURVE and based on the modal weighting coefficients in Table 5.10 as shown in Figure 5.14. The HOLOCURVE strain values oscillate about zero as one travels in the y direction and do not agree with the NASTRAN generated strains. Physically speaking, this anomalous result is due to the fact that it is the shearing stress or strain that is a maximum in the chordwise direction for a purely torsional mode shape, thus, making the principal stresses (and curvatures) a minimum. NASTRAN gives reasonable, albeit small, strain values because it relies on a two-dimensional plane stress theory while HOLOCURVE utilizes a simple beam theory. The saving grace in this instance is that the strain is small. In other words, along the plate directions where the bending strain amplitude is moderate-to-large, HOLOCURVE is readily applicable.

As a precaution against use of strain values based on insignificant modal weighting coefficients, HOLOCURVE determines the absolute value of all of the modal weighting coefficients in the beam series. If the absolute value of the largest coefficient is ten times greater than that of the next coefficient, a message is transmitted to the program operator. Further, if the largest coefficient corresponds to the translational or rotational series component in the free-free beam series, a message to this effort is transmitted to the user.

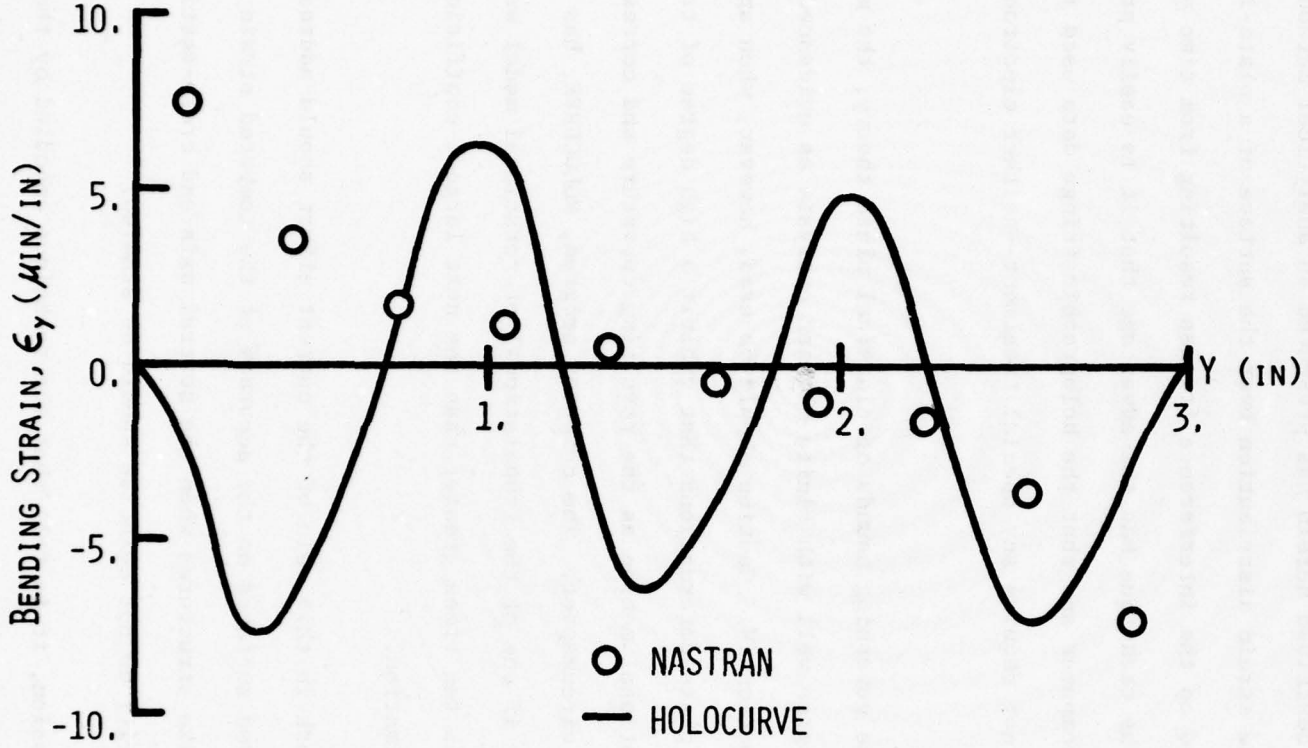


Figure 5.14 - Bending Strain, ϵ_y, vs. Chord, Y, at X=1.0" for Second Torsion Mode

SECTION VI

DISCUSSION AND CONCLUSION

The work described herein has presented an analytical method for determining the strain distribution over the surface of a plate-like structure based on the interference fringes resulting from time average holography. The technique has the advantage that it is easily programmed on a digital computer and that the holographic fringe data used in the solution does not require any special treatment--neither electronic nor optical.

Within the governing bounds of classical plate theory, the method was shown to agree well with finite element analysis as evidenced by the examples in Section IV. Caution should be used, however, when applying the method to plate deformations that exhibit a high degree of translational or rotational motion as the resulting curvature and corresponding strain may be circumspect. The computer program, HOLOCURVE, has a warning to this effect if one of the translational or rotational modal weighting coefficients is ten times greater than the next largest coefficient in the series approximation.

Future work in this vein of the current effort should address the effect of curved surfaces on the accuracy of the computed strain values. Also, plate-like structures when the neutral axis and cross-sectional center of gravity do not coincide should be studied.

In conclusion, it is felt that the technique embodied by the computer program, HOLOCURVE, will provide a useful tool for analyzing the surface strain distribution of vibration mode shapes of plate-like structures such as turbine engine blading.

APPENDIX A

PSEUDOINVERSE CALCULATION

Given a k x p matrix, A, one can compute its psudoinverse as follows:

1. Given:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad k = 3,\ p = 2 \qquad \text{(A-1)}$$

2. We can then compute A^TA.

$$A^TA = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \qquad \text{(A-2)}$$

3. Compute the p non-negative square roots of the eigenvalues, s, of A^TA to form the diagonal matrix, Σ.

$$\begin{vmatrix} 5-s & 1 \\ 1 & 5-s \end{vmatrix} = s^2 - 10s + 24 = 0 \qquad \text{(A-3)}$$

$$s_1 = 6,\ s_2 = 4$$

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{(A-4)}$$

4. Compute the orthonormalized eigenvectors, V_{iN}, of A^TA to form the matrix V_N.

$$(A^TA - s_i I)\, V_i = 0, \quad i = 1, 2 \tag{A-5}$$

For $s_1 = 6$:

$$\left\{\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right\} V_1 = 0 \tag{A-6}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{A-7}$$

Thus: $V_1 = \frac{\beta_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $V_{1N} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (A-8)

Similarly, for $s_2 = 4$:

$$V_2 = \frac{\beta_2}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } V_{2N} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{A-9}$$

Hence, from equations (A-8) and (A-9), we have:

$$V_N = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = V_N^T \tag{A-10}$$

5. Compute the p orthonormalized eigenvectors, U_{iN}, based on the p largest eigenvalues of AA^T to form the matrix U.

$$AA^T = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{bmatrix} \tag{A-11}$$

The eigenvalues of AA^T can be found to be:

$$s_1 = 6 \tag{A-12}$$

$$s_2 = 4$$

$$s_3 = 0$$

The two orthonormalized eigenvectors based on the largest two eigenvalues given in equation (A-12) can now be determined.

For $s_1 = 6$, (A-13)

$$U_1 = \frac{\beta_1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad U_{1N} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $s_1 = 4$,

$$U_2 = \frac{\beta_2}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad U_{2N} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{A-14}$$

Hence,

$$U_N = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{A-15}$$

6. The matrix, A, can be represented as,

$$A = U_N \Sigma V_N^T \qquad \text{(A-16)}$$

This can be checked by inserting equations (A-15), (A-4), and (A-10) into (A-16) to get:

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \qquad \text{Q.E.D.} \qquad \text{(A-16)}$$

7. The pseudoinverse of A is given as:

$$A+ = V_N \Sigma^{-1} U_N^T$$

Hence:

$$A+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & & \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad \text{(A-17)}$$

$$A+ = \frac{1}{12} \begin{bmatrix} 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix} \qquad \text{(A-18)}$$

It can be easily shown that equation (A-18) will satisfy the conditions for the pseudoinverse one of which is:

$$AA^+A = A \qquad \text{(A-19)}$$

APPENDIX B

HOLOCURVE LISTING

```
      PROGRAM TSL(DATA,PRINT,OUTPUT,INPUT,TAPE5=DATA,TAPE6=PRINT,
     1 TAPE7=INPUT,TAPE8=OUTPUT,LUNCH,TAPE10=LUNCH)
      COMMON/THICK/XTHK(20),YTHK(20)
      DIMENSION HKAREA(500)
      DIMENSION YDATN(40)
      DIMENSION XDATA(40),YDATA(40),YSTOR(3,40)
      DIMENSION X(64),Y(64),W(64)
      DIMENSION AMODE(20)
      DIMENSION AA(20,20),BB(20,1)
      COMMON/  /AA,BB
      EQUIVALENCE  (AA(1),YSTOR(1))
       INTEGER P,Q,PQ                                                    000190
      REAL NU                                                            000200
      COMMON/AZ/FX(20),FY(20),AMN(64 ),ALPH(20),BETA(20),P,Q,AL,BL,
     1 XMODE
      DIMENSION OPUT(7)
      DATA (OPUT(I),I=1,3)/6HW(X,Y),6H WXX  ,6H WYY  /
C                                                                        000190
C  MODE SHAPE FOR FIXED-FREE BEAM                                        000200
      XN(X,F,A,ALPHA)=          (1.-ALPHA)*EXP(F*X/A)/2.                 000210
     1 +(1.+ALPHA)*EXP(-F*X/A)/2. -COS(F*X/A) +ALPHA*SIN(F*X/A)          000220
C                                                                        000270
C  MODE SHAPE FOR FREE-FREE BEAM                                         000280
      YN(Y,F,B,BETA)=           (1.-BETA)*EXP(F*Y/B)/2.                  000290
     1 +(1.+BETA)*EXP(-F*Y/B)/2. +COS(F*Y/B) -BETA*SIN(F*Y/B)            000310
C                                                                        000350
      CALL INITT(30)
      REWIND 5                                                           000360
C  COMPUTE FREQUENCIES AND MODE CONSTANTS TO 1.E-12 ACCURACY             000370
      CALL FREQ                                                          000380
C                                                                        000440
      WRITE(8,9432 )
 9432 FORMAT(//////,
     1 9X,*<<<< PROGRAM HOLOCURVE - VERSION 23 MAY 77 1000 >>>>*//)
      WRITE(8,401)
  401 FORMAT(///9X,*ANSWER ALL QUESTIONS IN FLOATING POINT OR */
     19X,*ALPHAMERICALLY AS INDICATED*/)
      CALL TINPUT(I)
      CALL NEWPAG
      L=0
  402 FORMAT(F10.5)
  403 CONTINUE
      L=L+1
      WRITE(8,400) L
  400 FORMAT(15X,*DATA SET NUMBER *I2)
      WRITE(8,404)
  404 FORMAT(/9X,*WHAT IS THE MODE NAME (8H)?---*)
      READ(7,405) AMODE(L)
  405 FORMAT(A8)
      WRITE(8,406)
  406 FORMAT(/9X,*WILL OUTPUT BE BASED ON CLAMPED-FREE (CF) OR*/
     1 9X,*FREE-FREE (FF) BEAM SERIES?  TYPE CF OR FF.---*)
      READ(7,407) ANSWR
  407 FORMAT(A2)
      IF(ANSWR .EQ. 2HCF) Q=1
      IF(ANSWR .EQ. 2HFF) P=1
      WRITE(6,9871) ANSWR
```

```
      4071 FORMAT(/5X,*HOW MANY *A2* BEAM SERIES TERMS WILL BE USED?---*)
           READ(7,402)TERMS
   60      PQ=TERMS
           IF(Q .EQ. 1) P=TERMS
           IF(P .EQ. 1) Q=TERMS
           WRITE(8,4076)
      4076 FORMAT(/5X,*INPUT NUMBER (F.P.) OF FRINGE VALUES.---*)
   65      READ(7,402) AFRG
           MQ=AFRG
           SEG= 5HCHORD
           IF( Q .EQ. 1) SEG= 5HSPAN
           WRITE(8,4075) SEG
   70 4075 FORMAT(/5X,*INPUT *A5* LENGTH.---*)
           READ(7,402) ALGTH
           IF( P .EQ. 1) BL=ALGTH
           IF(Q .EQ. 1) AL=ALGTH
           IF(Q .EQ. 1) WRITE(8,409)
   75      IF(P .EQ. 1) WRITE(8,410)
      409 FORMAT(/5X,*INPUT CONSTANT CHORD(Y) LOCATION.---*)
      410 FORMAT(/5X,*INPUT CONSTANT SPAN(X) LOCATION.---*)
           IF(Q .EQ. 1) READ(7,402) YCHORD
           IF(P .EQ. 1) READ(7,402) XSPAN
   80      DO 411 J=1,MQ
           IF(Q .EQ. 1) Y(J)=YCHORD
      411 IF(P .EQ. 1) X(J)=XSPAN
C
           CALL NEWPAG
   85      WRITE(8,80)
       80 FORMAT(/5X,*WILL FRINGE OR DISPLACEMENT VALUES BE INPUT? TYPE "F"
          1 OR "D". ---*)
           READ(7,661) CVT
      661 FORMAT(A1)
   90      IF(Q .EQ. 1) WRITE(8,412)
      412 FORMAT(/5X,*INPUT PAIRS OF SPAN LOCATION,FRINGE VALUE DATA.*/
          1 5X,*FREE FIELD FORMAT WITH EACH VALUE SEPARATED BY COMMAS.*/
          2/5X,*DATA=*)
           IF(P .EQ. 1) WRITE(8,413)
   95      413 FORMAT(/5X,*INPUT PAIRS OF CHORD LOCATION,FRINGE VALUE DATA.*/
          1 5X,*FREE FIELD FORMAT WITH EACH VALUE SEPARATED BY COMMAS.*/
          2/5X,*DATA=*)
           IF(Q .EQ. 1) READ(7,*) (X(J),W(J),J=1,MQ)
           IF(P .EQ. 1) READ(7,*) (Y(J),W(J),J=1,MQ)
  100      ITHK=1
           IF(Q .EQ. 1 ) WRITE(8,6601)
           IF(P .EQ. 1 ) WRITE(8,6602)
      6601 FORMAT(/5X*WILL DISTANCE FROM NEUTRAL AXIS VS.  SPAN LOCATION BE*
          1/5X,* INPUT - YES OR NO?---*)
  105      6602 FORMAT(/5X*WILL DISTANCE FROM NEUTRAL AXIS VS. CHORD LOCATION BE*
          1/5X,* INPUT - YES OR NO?---*)
           READ(7,661) ANS
           IF(ANS .EQ. 1HN)  GO TO 590
           ITHK=2
  110      WRITE(8,662)
       662 FORMAT(/5X*HOW MANY (SEGMENT LOCATION/NEUTRAL AXIS DISTANCE) */
          1 5X*VALUES WILL BE INPUT (MAX OF 20)?---*)
           READ(7,402) ATHK
           NTHK=ATHK
```

```
      115           WRITE(6,603) NTHK
                  603 FORMAT(/5X*INPUT *I2,* PAIRS OF (SEGMENT LOCATION,NEUTRAL AXIS DIS
                     1TANCE) VALUES.*//3X*DATA=*)
                      READ(7,*) (XTHK(J),YTHK(J),J=1,NTHK)
                C
      120           690 CONTINUE
                      CALL NEWPAG
                      WRITE(6,8008)
                 8008 FORMAT(1X,*DO YOU WISH TO SEE INPUT DATA (YES OR NO)?... *)
                      READ(7,8009) ANSWR
      125             8009 FORMAT(A3)
                      IF(ANSWR .EQ. 3HNO ) GO TO 8010
                C  PRINT OUT DATA
                      WRITE(6,15) AMODE(L),P,Q
                     15 FORMAT(//5X,A8,* MODE FRINGE VALUES FOR (P,Q) =*,I1,*, *I1//)
      130             DO 155 I=1,MQ
                     155 WRITE(6,16)I,X(I),Y(I),W(I)
                      16 FORMAT(5X,*I=*I2,5X,*X,Y=*F7.4,2X,F7.4,5X,*F=*F5.2)
                      IF(ITHK .EQ. 2) WRITE(6,77)
                      77 FORMAT(//10X,*SEGMENT LOCATION - THICKNESS VALUES*/)
      135             DO 78 I=1,NTHK
                      78 IF(ITHK .EQ. 2) WRITE(6,79) I ,XTHK(I), YTHK(I)
                      79 FORMAT(5X*I= *I2,5X,*SEG. LOC. =*F7.4,5X,*THICKNESS =*F7.4)
                      CALL TINPUT(I)
                 8010 CONTINUE
      140             IF( CVT .EQ. 1HF)  CALL CONVRT(W,MQ)
                C                                                              000830
                C  COMPUTE PHI(I,J), J=1,2,...P, K=1,2,...Q FOR EACH W(I), I=1,2,....PQ.000840
                      DO 13 I=1,MQ
                      IF(P .EQ. 1) GO TO 642
      145             DO 137 J=1,P
                     137 AA(I,J)=XM(X(I),FX(J),AL,ALPH(J))
                C
                      GO TO 13
                C
      150           642 DO 136 K=1,Q
                      IF(K .GT. 2) GO TO 130
                      GO TO (131,132),K
                     130 AA(I,K)=YM(Y(I),FY(K),BL,BETA(K))
                      GO TO 136
      155             131 AA(I,K)=1.
                      GO TO 136
                     132 AA(I,K)=SQRT(3.)*(1.-2.*Y(I)/BL)
                     136 CONTINUE
                      13 CONTINUE
      160           C  SOLVE FOR THE MODAL WEIGHTING COEF. IN THE MATRIX EQUATION,  000770
                C                                                              000780
                C  X(MQ,P)*A(P) = W(MQ) OR Y(MQ,Q)*B(Q) = W(MQ)                 000790
                C                                                              000800
                C  USING THE LINEAR LEAST SQUARES EQ. SOLVER, "LLSQAR".
      165             DO 14 I=1,MQ
                      14 BB(I,1)= W(I)
                      CALL LLSQAR(AA,BB,MQ,PQ,1,20,20  ,AKAREA,IER)
                      DO 144 I=1,PQ                                            000870
                     144 AMN(I)= BB(I,1)
      170             CALL COMPAR( IFLAG,XMAG,TRIG)
                C                                                              000910
```

```
      CALL NEWPAG
      BEAM=2HCF
      IF(P.EQ.1) BEAM=2HFF
175   WRITE(6,9000) AMODE(L),L,BEAM
 9000 FORMAT(//5X,*COMPUTATION COMPLETED FOR *A3,* MODE.  DATA SET=*I2,
     1/5X,*DO YOU WISH TO SEE MODAL WEIGHTING COEFICIENTS FOR THE */
     25X,A2,* BEAM SERIES - YES OR NO?---*)
      READ(7,9001) ANSWR
180   9001 FORMAT(A3)
      IF(ANSWR .EQ. 3HNO )  GO TO 9003
      DO 9002 J=1,P
      DO 9002 K=1,Q
      INC= K+Q*(J-1)
185   9002 WRITE(6,9004) J,K,AMN(INC)
      9004 FORMAT(/6X*A(*I2,1X,I2,*)=* 1PE12.5)
      WRITE(6,9005)
      9005 FORMAT(//5X*WHERE A(P,1) = A(P) OR A(1,Q) = B(Q)*/)
      CALL TINPUT(I)
190   9003 CONTINUE
C  COMPUTE W(X,Y) FOR GIVEN VALUES OF X AND Y
      CALL NEWPAG
      WRITE(8,9007) AMODE(L)
      9007 FORMAT(//5X*DISPLACEMENT AND CURVATURE AS A FUNCTION OF*
195   1/5X,*POSITION WILL NOW BE LISTED AND PLOTTED FOR MODE *A3/
     25X,*HOW MANY DATA POINTS DO YOU WANT (MAX OF 40)?---*)
      READ(7,9008) PTS
      9008 FORMAT(F10.5)
C
200   IF(P .EQ. 1) GO TO 9009
      AINC= AL/(PTS-1.)
      XO= -AINC
      GO TO 9011
C
205   9009 CONTINUE
      AINC=BL/(PTS-1.)
      YO= -AINC
C
      9011 IPTS=PTS
210   DO 19 I=1,IPTS
      IF(P .EQ.1)       YO= YO+AINC
      IF(Q .EQ.1)       XO= XO+AINC
      IF(Q .EQ. 1) YO=YCHORD
      IF(P .EQ.1) XO=XSPAN
215   CALL WDISP(XO,YO,WD)
      CALL CURV(XO,YO,WX2,WY2,ITHK,NTHK)
C
      IF( P .EQ. 1)     XDATA(I)= YO
      IF(Q .EQ. 1) XDATA(I)=XO
220   YSTOR(1,I)=WD
      IF(Q .EQ.1) YSTOR(2,I)= WX2
      IF(P .EQ.1) YSTOR(3,I)= WY2
   19 CONTINUE
C
225   CALL NEWPAG
C  DISPLACEMENT AND CURVATURE OUTPUT**********
      DO 9018 L=1,2
      NUM=1
```

```
      IF(LL .EQ.2 .AND. Q .EQ. 1) NUM=2
230   IF(LL .EQ.2 .AND. P .EQ. 1) NUM=3
      IF(LL .EQ. 1) WRITE(6,7050) AMODE(L)
 7050 FORMAT(//6X,*NORMAL DISPLACEMENT FOR MODE *A8//)
      IF(LL .EQ. 2) WRITE(6,7051) AMODE(L)
 7051 FORMAT(//6X,*CURVATURE FOR MODE *A8//)
235   IF(LL .EQ. 2 .AND. ITHK .EQ. 1) WRITE(6,9071)
 9071 FORMAT(/9X*(CURVATURE BASED ON UNIT THICKNESS)*/)
      IF(LL .EQ. 2 .AND. ITHK .EQ. 2) WRITE(6,9072)
 9072 FORMAT(/9X*(CURVATURE IS EQUIVALENT TO STRAIN)*/)
      WRITE(6,9015) OPUT(NUM),OPUT(NUM)
240   9015 FORMAT(//9X,*X*7X,*Y*,6X,A6,4X,*NORMALIZED *A6/)
      DO 9016 I=1,IPTS
      YDATA(I)=YSTOR(NUM,I)
      CALL DNORM(YDATA,YDATN,IPTS)
C
245   IF(Q.EQ.1)WRITE(6,9017)(XDATA(I),YO,YDATA(I),YDATN(I),I=1,IPTS)
      IF(P.EQ.1)WRITE(6,9017)(XO,XDATA(I),YDATA(I),YDATN(I),I=1,IPTS)
 9017 FORMAT(9X,0P,F6.3,2X,F6.3,2X,1PE12.5,0P,3X,F10.6)
      IF(Q .EQ. 1 .AND. TRIG .EQ. 1.)  GO TO 1001
      IF(P .EQ. 1 .AND. TRIG .EQ. 1.)  GO TO 1001
250   GO TO 1004
 1001 IF(IFLAG .EQ. 1) WRITE(6,1002) XMAG
      IF(IFLAG .EQ. 2) WRITE(6,1003) XMAG
      IF(IFLAG .GT. 2)  WRITE(6,1010) IFLAG,XMAG,IFLAG
 1002 FORMAT(//5X*TRANSLATIONAL FF BEAM SERIES COMPONENT, B(1),*
255     1 /5X*IS* 1PE10.3,* TIMES THE NEXT LARGEST SERIES TERM.*/
        2 5X,*CURVATURE  VALUES MUST BE VIEWED WITH CAUTION.*/)
 1003 FORMAT(//5X*   ROTATIONAL FF BEAM SERIES COMPONENT, B(2),*
        1 /5X*IS* 1PE10.3,* TIMES THE NEXT LARGEST SERIES TERM.*/
        2 5X,*CURVATURE  VALUES MUST BE VIEWED WITH CAUTION.*/)
260   1010 FORMAT(//5X,*FF SERIES TERM, B(*I2,*) IS* 1PE10.3,* TIMES*/
        1 5X,* THE NEXT LARGEST SERIES TERM INDICATING A STRONG*/
        2 5X,I2,* FLEXURE COMPONENT.*/)
      GO TO 1004
 1005 WRITE(6,1007) IFLAG,XMAG,IFLAG
265   1007 FORMAT(//5X,*OF SERIES TERM, A(*I2,*) IS* 1PE10.3,* TIMES*/
        1 5X,* THE NEXT LARGEST SERIES TERM INDICATING A STRONG*/
        2 5X,I2,* FLEXURE COMPONENT.*/)
 1004 CONTINUE
      CALL TINPUT(I)
270   CALL NEWPAG
C
      CALL PLTOAT(IPTS,XDATA,YDATA,NUM, P, Q)
C
      CALL NEWPAG
275   9018 CONTINUE
      P=1 & Q=1 & PQ=1
      WRITE(6,526)
  526 FORMAT(//5X*IS THERE MORE DATA TO BE ANALYZED (YES OR NO)?---*)
      READ(7,527) ANS
280   527 FORMAT(A1)
      IF(ANS .EQ. 1HN) STOP
      GO TO 403
      END
```

```
      SUBROUTINE WDISP(X,Y,WD    )                                          002500
C                                                                           002510
C  SUBR WDISP  COMPUTES THE NORMAL DISPLACEMENT AT POINT (X,Y)              002520
C  BASED ON THE SERIES EXPRESSION FOR W(X,Y).                               002530
C                                                                           002540
      INTEGER Q,P,PQ                                                        002550
      COMMON/AZ/FX(20),FY(20),AMN(60 ),ALPH(20),BETA(20),P,Q,AL,BL,         0025
     1 XMODE                                                                002570
C                                                                           002590
C                                                                           002620
      XM(X,F,A,ALPHA)=          (1.-ALPHA)*EXP(F*X/A)/2.                    002630
     1 +(1.+ALPHA)*EXP(-F*X/A)/2. -COS(F*X/A) +ALPHA*SIN(F*X/A)             002640
      YN(Y,F,B,BETA)=           (1.-BETA)*EXP(F*Y/B)/2.                     002650
     1 +(1.+BETA)*EXP(-F*Y/B)/2. +COS(F*Y/B) -BETA*SIN(F*Y/B)               002660
C                                                                           002670
      WD=0.                                                                 002680
      IF(P .EQ. 1) GO TO 642
      DO 137 J=1,P
      PHI    =XM(X,FX(J),AL,ALPH(J))
  137 WD=WD+AMN( J )*PHI
      GO TO 10
  642 DO 138 K=1,Q
      IF(K .GT. 2) GO TO 130
      GO TO (131,132),K
  130 PHI    =YN(Y,FY(K),BL,BETA(K))
      GO TO 138
  131 PHI=1.
       GO TO 138
  132 PHI= SQRT(3.)*(1.-2.*Y/BL)
  138 WD= WD+AMN(K)*PHI
   10 CONTINUE
      RETURN                                                                002800
      END                                                                   002810
```

```
      SUBROUTINE CURV(X,Y,EX,EY,ITHK,NTHK)
      COMMON/THICK/XTHK(20),YTHK(20)
      DIMENSION TEMP(6)
C                                                                      003360
      INTEGER P,Q,PQ                                                   003370
      COMMON/AZ/FX(20),FY(20),AMN(40 ),ALPH(20),BETA(20),P,Q,AL,BL,    0033
     1 XMODE                                                           003390
C                                                                      003410
C                                                                      003420
      XM(X,F,A,ALPHA)=          (1.-ALPHA)*EXP(F*X/A)/2.               003430
     1 +(1.+ALPHA)*EXP(-F*X/A)/2. -COS(F*X/A) +ALPHA*SIN(F*X/A)        003440
      X1(X,F,A,ALPHA)=(F/A) *(  (1.-ALPHA)*EXP(F*X/A)/2.               003450
     1 -(1.+ALPHA)*EXP(-F*X/A)/2. +SIN(F*X/A) +ALPHA*COS(F*X/A))       003460
      X2(X,F,A,ALPHA)=(F/A)**2*((1.-ALPHA)*EXP(F*X/A)/2.               003470
     1 +(1.+ALPHA)*EXP(-F*X/A)/2. +COS(F*X/A) -ALPHA*SIN(F*X/A))       003480
C                                                                      003490
      YM(Y,F,B,BETA)=           (1.-BETA)*EXP(F*Y/B)/2.                003500
     1 +(1.+BETA)*EXP(-F*Y/B)/2. +COS(F*Y/B) -BETA*SIN(F*Y/B)          003510
      Y1(Y,F,B,BETA)=(F/B) *(  (1.-BETA)*EXP(F*Y/B)/2.                 003520
     1 -(1.+BETA)*EXP(-F*Y/B)/2. -SIN(F*Y/B) -BETA*COS(F*Y/B))         003530
      Y2(Y,F,B,BETA)=(F/B)**2*((1.-BETA)*EXP(F*Y/B)/2.                 003540
     1 +(1.+BETA)*EXP(-F*Y/B)/2. -COS(F*Y/B) +BETA*SIN(F*Y/B))         003550
C                                                                      003560
C  COMPUTE STRAINS                                                     003570
      EX=0.                                                            003600
      EY=0.                                                            003610
      IF(P .EQ. 1) GO TO 642
      DO 137 J=1,P
      PEX= X2(X,FX(J),AL,ALPH(J))
  137 EX=EX+AMN(J)*PEX
      GO TO 10
  642 DO 138 K=1,Q
      IF(K .GT. 2) GO TO 130
      GO TO (131,132),K
  130 PEY=Y2(Y,FY(K),BL,BETA(K))
      GO TO 138
  131 PEY=0.
      GO TO 138
  132 PEY=0.
  138 EY=EY+AMN(K)*PEY
C
   10 Z=1.
      IF(ITHK .EQ. 1) GO TO 50
      IF(P .EQ. 1 .AND. ITHK .EQ. 2) CALL INTERP(XTHK,YTHK,NTHK,2,Y,Z,
     1 TEMP,IER)
      IF(Q.EQ.1.AND.ITHK.EQ.2) CALL INTERP(XTHK,YTHK,NTHK,2,X,Z,
     1 TEMP,IER)
      IF (P .EQ. 1) EY= Z/2.*EY
      IF( Q .EQ. 1) EX= Z/2.*EX
   50 RETURN
      END                                                              004140
```

```
      SUBROUTINE FREQ                                                   004100
      COMMON/AZ/FX(20),FY(20),AMN(6, ),ALPH(20),BETA(20),P,Q,AL,BL,     0025
     1  XMODE                                                          004170
C DEFINE COSH AND SINH FUNCTIONS                                        004180
      COSH(X)= (EXP(X)+EXP(-X))/2.                                      004190
      SINH(X)= (EXP(X)-EXP(-X))/2.                                      004200
C  CHARACTERISTIC EQS. AND THEIR DERIVATIVES                            004210
      FF(X)= COS(X)*COSH(X) - 1.                                        004220
      FP(X)= -SIN(X)*COSH(X) + COS(X)*SINH(X)                           004230
      GG(X)= COS(X)*COSH(X) + 1.                                        004240
      GP(X)= FP(X)                                                      004250
C  FIXED-FREE BEAM FREQUENCIES                                          004260
      DATA (FX(I),I=1,6)/ 1.8751041, 4.6940911, 7.8547574,              004270
     1  10.995541, 14.1371684, 17.2787597 /                            004280
C  FREE-FREE BEAM FREQUENCIES                                           004290
      DATA (FY(I),I=3,6)/ 4.7300408, 7.8532046, 10.9956078,14.1371655/  004300
      DO 299 I=7,20                                                     004310
      PI= 3.1415926535                                                  004320
      AI=I                                                              004330
      FX(I)= (2.*AI-1.)*PI/2.                                           004340
  299 FY(I)= (2.*AI-3.)*PI/2.                                           004350
      ERROR= 1.E-12                                                     004360
      DO 25 I=1,20                                                      004370
      RN= FX(I)                                                         004380
C                                                                       004390
   27 RN1= RN - GG(RN)/GP(RN)                                           004400
      IF( ABS(RN1-RN) .LT. ERROR)  GO TO 28                             004410
      RN= RN1                                                           004420
      GO TO 27                                                          004430
C                                                                       004440
   28 FX(I)= RN                                                         004450
      ALPH(I)= (COS(RN)+COSH(RN))/(SIN(RN)+SINH(RN))                    004460
      IF( I .LE. 2) GO TO 26                                            004470
      RN= FY(I)                                                         004480
   29 RN1= RN - FF(RN)/FP(RN)                                           004490
      IF( ABS(RN1-RN) .LT. ERROR)  GO TO 30                             004500
      RN= RN1                                                           004510
      GO TO 29                                                          004520
C                                                                       004530
   30 FY(I)= RN                                                         004540
      BETA(I)= (COS(RN) - COSH(RN))/(SIN(RN) - SINH(RN))                004550
   25 CONTINUE                                                          004560
      RETURN                                                            004570
      END                                                               004580
```

```
      SUBROUTINE PLTDAT(IPTS,XDATA,YDATA,NUM,P,Q)
      DIMENSION IW(14),IWXX(13),IWYY(13),IX(8),IY(9)
      DIMENSION XDATA(IPTS),YDATA(IPTS)
      INTEGER P,Q
C
      DATA IW/87,32,66,73,83,80,76,65,87,69,77,66,73,84/
C
      DATA IWXX/87,66,80,82,67,65,32,86,65,84,66,82,69/
      DATA IWYY/87,69,89,32,67,65,82,86,65,84,66,82,69/
C
      DATA IX/66,32,46,63,80,65,78,41/
      DATA IY/89,32,46,67,72,79,82,68,41/
C
      CALL INITT(30)
        CALL BINITT
      CALL NPTS(IPTS)
      CALL CHECK(XDATA,YDATA)
      CALL SYMBL(1)
      CALL DSPLAY(XDATA,YDATA)
      CALL MOVABS(500,32)
      IF(P .EQ. 1) CALL HLABEL(9,IY)
      IF(Q .EQ. 1) CALL HLABEL(8,IX)
C
      CALL MOVABS(48,630)
      IF(NUM .EQ. 1) CALL VLABEL(14,IW)
      IF(NUM .EQ. 2) CALL VLABEL(13,IWXX)
      IF(NUM .EQ. 3) CALL VLABEL(13,IWYY)
      CALL TINPUT(I)
      CALL FINITT(0,760)
      RETURN
      END
```

```
      SUBROUTINE CONVRT(W,N)                                            000010
      DIMENSION W(N),BESO(99)                                           000020
      BESO(1)=2.40482555                                                000025
      PI=3.14159265                                                     000030
      DO 10 I=2,99                                                      000040
      A=4.*FLOAT(I)-1.                                                  000050
      P=PI*A                                                            000060
   10 BESO(I)= .25*P*(1.+2./P**2- 62./3./P**4                           000070
     1 +15116./15./P**6 - 12554.74./105./P**8                           000080
     2 +8366629.292./315./P**10)                                        000090
      XLAM=6326.E-6/2.5399                                              000100
      DO 11 I=1,N                                                       000110
      K=IFIX(ABS(W(I)))                                                 000120
      IF(K .EQ. 0) GO TO 11                                             000125
      SGN=1.                                                            000130
      IF(W(I) .LT. 0.) SGN=-1.                                          000140
      W(I)=SGN*XLAM*BESO(K)/(4.*PI)                                     000150
   11 IF(W(I) .EQ. 0.) W(I)=0.                                          000160
      RETURN                                                            000170
      END                                                               000180
```

```
      SUBROUTINE COMPAR(IFLAG1,XMAG,TRIG)
C   THIS SUB-R COMPARES THE BEAM SERIES TERMS WITH EACH OTH R TO SEE IF
C   ONE TERM IS APPRECIABLY LARGER THAN THE OTHERS.  THIS IS PARTICULARLY
C   IMPORTANT FOR THE CASE OF THE FREE-FREE BEAM SERIES WHERE THE
C   TRANSLATIONAL OR ROTATIONAL MODES CAN DOMINATE GIVING RISE TO
C   ERRONEOUS SECOND DERIVATIVES.
C
      INTEGER P,Q,PQ
      COMMON/AZ/FX(20),FY(20),AMN(64 ),ALPH(20),BETA(20),P,Q,AL,BL,     0025
     1 XMODE
      PQ=P*Q
C   FIND LARGEST AMN
      BIG1=0.
      DO 10 I=1,PQ
      IF( ABS(AMN(I)) .GT. BIG1)  GO TO 9
      GO TO 10
    9 IFLAG1=I
      BIG1= ABS(AMN(IFLAG1))
   10 CONTINUE
C
C   FIND SECOND LARGEST BEAM SERIES TERM
      BIG2=0.
      DO 11 I=1,PQ
      IF( I .EQ. IFLAG1) GO TO 11
      IF( ABS(AMN(I)) .GT. BIG2) GO TO 12
      GO TO 11
   12 IFLAG2=I
      BIG2= ABS(AMN(IFLAG2))
   11 CONTINUE
C
      XMAG= BIG1/BIG2
      TRIG=0.
      IF(XMAG .GT. 10.) TRIG=1.
      RETURN
      END
```

```
      C
            SUBROUTINE DNORM(YD,YN,IPTS)
            DIMENSION YD(400),YN(400)
      C
      C   FIND LARGEST YD VALUE
            BIG=0.
            DO 10 I=1,IPTS
         10 IF(ABS(YD(I)) .GT. BIG)  BIG=ABS(YD(I))
      C  NORMALIZE YD DATA
            DO 11 I=1,IPTS
         11 YN(I)= YD(I)/BIG
            RETURN
            END
```

```
 CSA    NOS/BE CYBER        6600 CYR1 03/28/77
13.55.09.JMDAE3W  FROM    /AE
13.55.09.IP  J0002792 WORDS - FILE INPUT  , DC 00
13.55.09.JN,T10,1012,CM105000,STCSA.P720630  MACB
13.55.09.AI 1 32081 NEW CY=1
13.55.12.FT 1.
13.55.36.LOCKIN.
13.55.42.      3.673 CP SECONDS COMPILATION TIME
13.55.42.ATTACH,CC6600,ID=X654321,SN=ASD.
13.55.42.PFN IS
13.55.42.CC6600
13.55.42.PF CYCLE NO. = 001
13.55.42.ATTACH,TEKLIB,TEK.RONIXLIB,ID=P720612.
13.55.43.PF CYCLE NO. = 999
13.55.43.ATTACH,IMSL,IMSL,ID=X654321,SN=ASD.
13.55.43.PF CYCLE NO. = 001
13.55.43.LIBRARY,TEKLIB,IMSL.
13.55.43.REQUEST,ABS,*PF.
13.55.46.MAP,PART.
13.55.46.LDSET,LIB=CC6600.
13.55.47.LOAD,LGO.
13.55.47.NOGO,ABS.
14.00.07.PURGE,LF1,NEW,CY=1.
14.00.08.PR ID=  P720630 PFN=NEW
14.00.08.PR CY= 001  0002368 WORDS.
14.00.08.CATALOG,ABS,NEW,CY=1,RP=50.
14.00.08.INITIAL CATALOG
14.00.09.CT ID=  P720630 PFN=NEW
14.00.09.CT CY= 001  0002368 WORDS.
14.00.09.OP  J0008768 WORDS - FILE OUTPUT , DC 40
14.00.09.MS   46592  WORDS  (   46592 MAX USED)
14.00.09.SCM  7+000  WORDS MAXIMUM
14.00.09.CPA       6.592 SEC.          2.905 ADJ.
14.00.09.IO        3.500 SEC.          4.752 ADJ.
14.00.09.CM      383.+37 KWS.          2.+47 ADJ.
14.00.09.CRUS                         10.636
14.00.09.COST                $          .63
14.00.09.PP       35.610 SEC.      DATE 05/23/77
14.00.09.EJ  END OF JOB, AE  P720630.

**********        JMDAE3W   //// END  OF LIST ////
```

APPENDIX C

HOLOCURVE EXAMPLE

```
ASD COMPUTER CENTER INTERCOM 4.5
SYSTEM  CSA
DATE  06/01/77       TIME  10.50.12.

PLEASE LOGIN
LOGIN,P720611,BB,305


06/01/77   LOGGED IN AT  10.50.35.
           WITH USER-ID JR
           EQUIP/PORT 16/031
COMMAND- ATTACH,HOLO,NEW,CY=1,ID=P720630

COMMAND- HOLO
```

<<<< PROGRAM HOLOCURVE - VERSION 23 MAY 77 1000 >>>>

ANSWER ALL QUESTIONS IN FLOATING POINT OR
ALPHAMERICALLY AS INDICATED

77

```
                DATA SET NUMBER  1
WHAT IS THE MODE NAME (8H)?---2 FLEX

WILL OUTPUT BE BASED ON CLAMPED-FREE (CF) OR
FREE-FREE (FF) BEAM SERIES?  TYPE CF OR FF.---CF

HOW MANY CF BEAM SERIES TERMS WILL BE USED?---5.

INPUT NUMBER (F.P.) OF FRINGE VALUES.---7.

INPUT SPAN  LENGTH.---7.

INPUT CONSTANT CHORD(Y) LOCATION.----.5
```

```
WILL FRINGE OR DISPLACEMENT VALUES BE INPUT? TYPE "F"  OR "D". ---F

INPUT PAIRS OF SPAN LOCATION,FRINGE VALUE DATA.
FREE FIELD FORMAT WITH EACH VALUE SEPARATED BY COMMAS.

DATA=1.,-5.,2.,-11.,3.,-12.,4.,-8.,4.94,0.,6.,9.,7.,16.


WILL DISTANCE FROM NEUTRAL AXIS VS.  SPAN LOCATION BE
 INPUT - YES OR NO?---NO
```

DO YOU WISH TO SEE INPUT DATA (YES OR NO)?... YES

2 FLEX MODE FRINGE VALUES FOR (P,Q) =5, 1

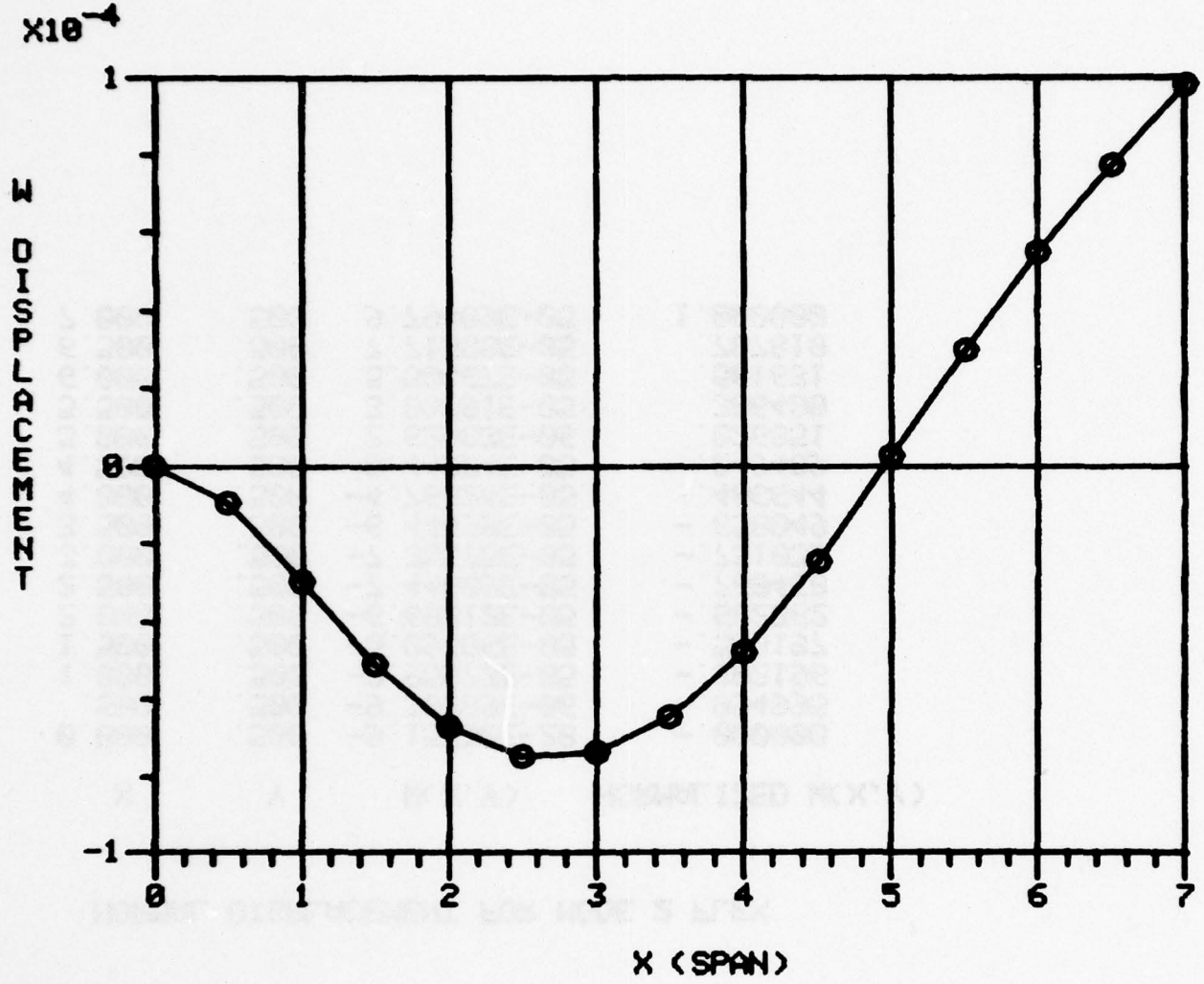
I= 1	X,Y= 1.0000	.5000	F= -5.00
I= 2	X,Y= 2.0000	.5000	F=-11.00
I= 3	X,Y= 3.0000	.5000	F=-12.00
I= 4	X,Y= 4.0000	.5000	F= -8.00
I= 5	X,Y= 4.9400	.5000	F= 0.00
I= 6	X,Y= 6.0000	.5000	F= 9.00
I= 7	X,Y= 7.0000	.5000	F= 16.00

```
COMPUTATION COMPLETED FOR 2 FLEX   MODE.  DATA SET= 1
DO YOU WISH TO SEE MODAL WEIGHTING COEFICIENTS FOR THE
CF BEAM SERIES - YES OR NO?---YES

 A( 1  1)= 1.17223E-05

 A( 2  1)=-5.08400E-05

 A( 3  1)=-1.25548E-05

 A( 4  1)=-4.61346E-07

 A( 5  1)=-1.49835E-06

WHERE A(P,1) = A(P) OR A(1,Q) = B(Q)
```

DISPLACEMENT AND CURVATURE AS A FUNCTION OF
POSITION WILL NOW BE LISTED AND PLOTTED FOR MODE 2 FLEX
HOW MANY DATA POINTS DO YOU WANT (MAX OF 40)?---15.

NORMAL DISPLACEMENT FOR MODE 2 FLEX

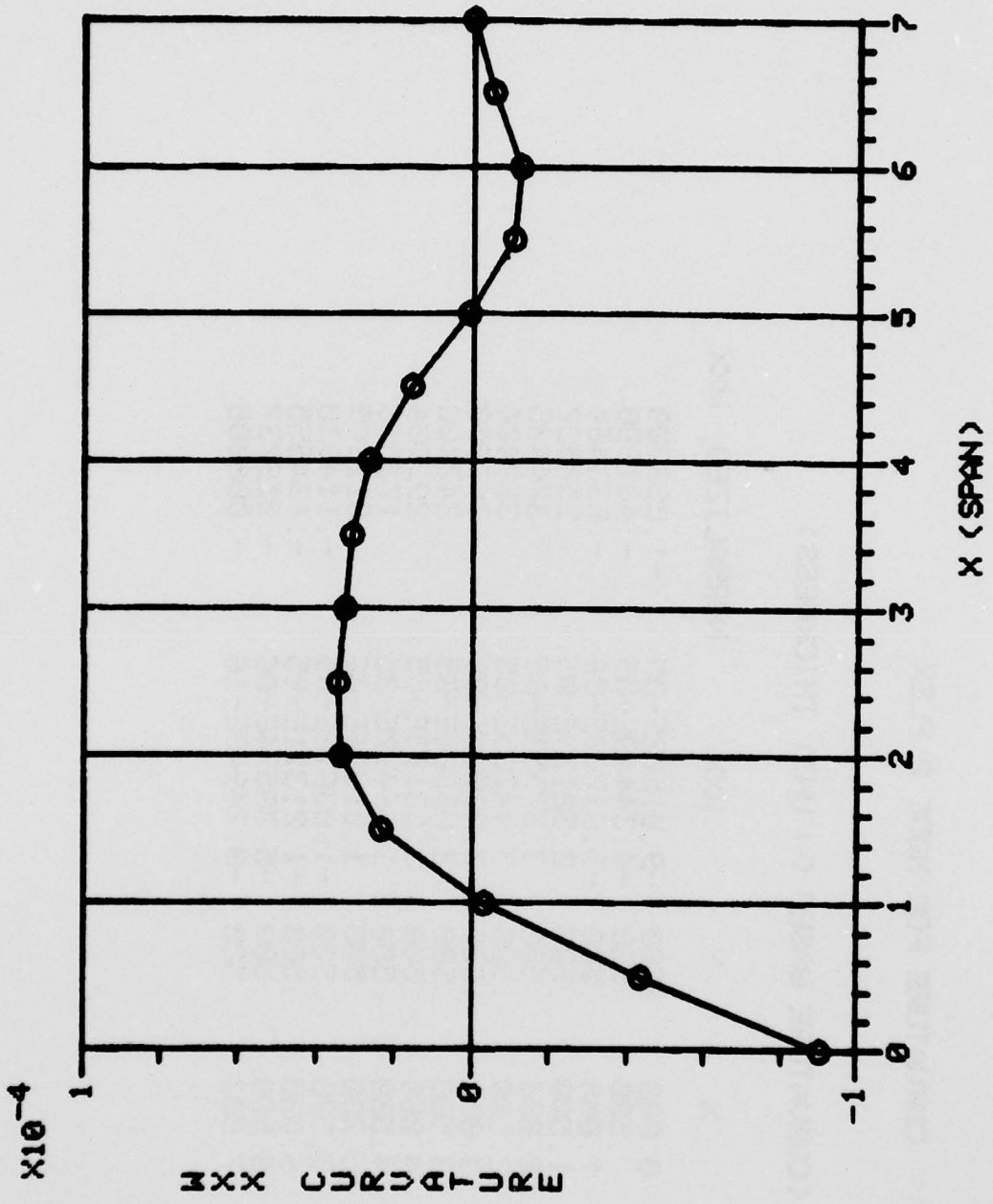
X	Y	W(X,Y)	NORMALIZED W(X,Y)
0.000	.500	-9.15730E-20	-.000000
.500	.500	-9.30389E-06	-.094995
1.000	.500	-2.95973E-05	-.302196
1.500	.500	-5.09486E-05	-.520197
2.000	.500	-6.68312E-05	-.682362
2.500	.500	-7.44809E-05	-.760468
3.000	.500	-7.35575E-05	-.751039
3.500	.500	-6.44499E-05	-.658049
4.000	.500	-4.76624E-05	-.486644
4.500	.500	-2.44274E-05	-.249409
5.000	.500	2.62983E-06	.026851
5.500	.500	3.00091E-05	.306400
6.000	.500	5.50067E-05	.561631
6.500	.500	7.71588E-05	.787810
7.000	.500	9.79409E-05	1.000000


X10⁻⁴
1
0
-1
W DISPLACEMENT
0
1
2
3
4
5
6
7
X (SPAN)

CURVATURE FOR MODE 2 FLEX

(CURVATURE BASED ON UNIT THICKNESS)

X	Y	WXX	NORMALIZED WXX
0.000	.500	-9.01572E-05	-1.000000
.500	.500	-4.34449E-05	-.481880
1.000	.500	-3.01700E-06	-.033464
1.500	.500	2.32179E-05	.257527
2.000	.500	3.37553E-05	.374405
2.500	.500	3.44765E-05	.382404
3.000	.500	3.27090E-05	.362800
3.500	.500	3.09600E-05	.343400
4.000	.500	2.63173E-05	.291904
4.500	.500	1.56437E-05	.173516
5.000	.500	1.01384E-06	.011245
5.500	.500	-1.03618E-05	-.114930
6.000	.500	-1.21311E-05	-.134555
6.500	.500	-5.38574E-06	-.059737
7.000	.500	-9.74940E-16	-.000000


MAGNIFIED XXR
X10⁻⁴
1
0
-1
0
1
2
3
4
5
6
7
X (SPAN)

```
    IS THERE MORE DATA TO BE ANALYZED (YES OR NO)?---NO

     STOP
        .436 CP SECONDS EXECUTION TIME
COMMAND- LOGOUT

CPA       2.771 SEC.          1.202 ADJ.
CRUS                          2.676
CONNECT TIME    0 HRS.    14 MIN.
 06/01/77  LOGGED OUT AT 11.04.06.
       <
```

APPENDIX D

LLSQAR DESCRIPTION

AD-A046 153 AIR FORCE AERO PROPULSION LAB WRIGHT-PATTERSON AFB OHIO F/G 20/11
QUANTITATIVE DISPLACEMENT AND STRAIN DISTRIBUTION OF VIBRATING --ETC(U)
JUL 77 J C MACBAIN

UNCLASSIFIED AFAPL-TR-77-44 NL

2 OF 2
AD A046153

END
DATE FILMED
12-77
DDC

```
      SUBROUTINE LLSQAR (A,B,M,NA,NB,IA,IB,IDGT,WKAREA,IER)                LLSQ0010
C                                                                          LLSQ0020
C-LLSQAR--------S-------LIBRARY 3----------------------------------------LLSQ0030
C                                                                          LLSQ0040
C   FUNCTION              - LEAST SQUARES SOLUTION OF OVERDETERMINED       LLSQ0050
C                             SYSTEM OF LINEAR EQUATIONS                   LLSQ0060
C   USAGE                 - CALL LLSQAR(A,B,M,NA,NB,IA,IB,IDGT,WKAREA,IER)LLSQ0070
C   PARAMETERS   A        - THE COEFFICIENT MATRIX OF THE EQUATION         LLSQ0080
C                             AX=B, WHERE A IS M X NA WITH M GREATER THAN  LLSQ0090
C                             OR EQUAL TO NA. INPUT A IS REPLACED BY       LLSQ0100
C                             THE PSEUDO-INVERSE OF A                      LLSQ0110
C                B        - MATRIX OF THE RIGHT HAND SIDE OF THE EQUATION LLSQ0120
C                             AX=B, WHERE B IS M X NB. THE NA X NB         LLSQ0130
C                             SOLUTION X OVERWRITES B.                     LLSQ0140
C                M        - NUMBER OF ROWS IN A AND B.                    LLSQ0150
C                NA       - NUMBER OF COLUMNS IN MATRIX A                 LLSQ0160
C                NB       - NUMBER OF COLUMNS IN MATRIX B                 LLSQ0170
C                IA       - ROW DIMENSION OF A IN THE CALLING             LLSQ0180
C                             PROGRAM.                                     LLSQ0190
C                IB       - ROW DIMENSION OF B IN THE CALLING             LLSQ0200
C                             PROGRAM.                                     LLSQ0210
C                IDGT     - THE ELEMENTS OF A ARE ASSUMED TO BE CORRECT   LLSQ0220
C                             TO IDGT SIGNIFICANT  DIGITS. IDGT IS AN      LLSQ0230
C                             INPUT PARAMETER.                             LLSQ0240
C                WKAREA   - WORK AREA OF DIMENSION GREATER THAN OR EQUAL  LLSQ0250
C                             TO NA(NA+4).                                 LLSQ0260
C                IER      - ERROR PARAMETER                               LLSQ0270
C                             TERMINAL ERROR = 128 + N                     LLSQ0280
C                               N = 1  INDICATES THAT INPUT A IS A ZERO    LLSQ0290
C                                      MATRIX.                             LLSQ0300
C   PRECISION             - SINGLE                                         LLSQ0310
C   REQD. IMSL ROUTINES   - LPSDOR,LSVALR,UERTST,VSORTM                    LLSQ0320
C   LANGUAGE              - FORTRAN                                        LLSQ0330
C----------------------------------------------------------------------LLSQ0340
```

CALL LLSQAR (A,B,M,NA,NB,IA,IB,IDGT,WKAREA,IER)

Purpose

LLSQAR performs the least squares solution of an overdetermined system of linear equations. Given the M by NA matrix A and the M by NB matrix B, where M is not less than NA, this routine finds the NA by NB matrix X such that $||X||_E$ and $||AX-B||_E$ = minimum, where $||\cdot||_E$ is the Euclidean norm of a matrix. That is,

$$||AX-B||_E = \left[\sum_{i=1}^{M}\sum_{j=1}^{NB}\left(\sum_{k=1}^{NA} A_{ik}X_{kj}-B_{ij}\right)^2\right]^{1/2}$$

Algorithm

The routine first calculates the pseudo-inverse of the matrix A by calling IMSL routine LPSDOR. $A^+ = V(S^+)U^T$. (See the documentation of LPSDOR.) V is an NA by NA orthogonal matrix, S^+ is an NA by NA diagonal matrix and U^T is an NA by M matrix with orthogonal rows. The solution of the equations is computed by $V(S^+)U^TB$.

Programming Notes

1. In the calling program, both A and B are two-dimensional arrays. The value of the first dimension of A is IA(IA not less than M), and the value of the first dimension of B is IB(IB not less than M).

2. The work area, WKAREA, must be a one-dimensional array of length at least NA(NA+4).

3. The input matrix A is replaced by the pseudo-inverse of A.

4. The NA by NB solution X overwrites input matrix B.

Accuracy

The elements of A are assumed to be correct to IDGT significant digits. The routine computes the pseudo-inverse of $\overline{A} = USV^T$; $\overline{A}$ differs from A in the Euclidean norm by less than $d = \sqrt{M*NA} * (\max_{i,j} a_{ij}) * 10^{-IDGT}$, and of all matrices which differ from A by less than d in the Euclidean norm, $\overline{A}$ has minimal rank.

Example

```
DIMENSION A(6,3),B(7,4),WKAREA(60)
```

Input:

```
M    = 6
NA   = 3
NB   = 4
IA   = 6
IB   = 7
IDGT = 4
```

$$A = \begin{bmatrix} 33.0 & 16.0 & 72.0 \\ -24.0 & -10.0 & -57.0 \\ -8.0 & -4.0 & -17.0 \\ 33.0 & 16.0 & 72.0 \\ -24.0 & -10.0 & -57.0 \\ -8.0 & -4.0 & -17.0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -359.0 \\ 0.0 & 1.0 & 0.0 & 281.0 \\ 0.0 & 0.0 & 1.0 & 85.0 \\ 1.0 & 0.0 & 0.0 & -359.0 \\ 0.0 & 1.0 & 0.0 & 281.0 \\ 0.0 & 0.0 & 1.0 & 85.0 \\ x & x & x & x \end{bmatrix}$$

```
CALL LLSQAR (A,B,M,NA,NB,IA,IB,IDGT,WKAREA,IER)
```

Output:

$$X = \begin{bmatrix} -9.6667 & -2.6667 & -32.000 & 1.0000 \\ 8.0000 & 2.5000 & 25.500 & -2.0000 \\ 2.6667 & .66667 & 9.0000 & -5.0000 \end{bmatrix}$$

Note: x denotes an element which is not used by LLSQAR.

```
      SUBROUTINE LPSDOR (A,M,N,IA,AINV,IDGT,WKAREA,IER)                  LPSD0010
C                                                                        LPSD0020
C-LPSDOR--------S-------LIBRARY 3----------------------------------------LPSD0030
C                                                                        LPSD0040
C   FUNCTION              - PSEUDO-INVERSE OF A MATRIX.                  LPSD0050
C   USAGE                 - CALL LPSDOR (A,M,N,IA,AINV,IDGT,WKAREA,IER)  LPSD0060
C   PARAMETERS   A        - A GIVEN MATRIX OF SIZE M X N                 LPSD0070
C                M        - NUMBER OF ROWS IN MATRIX A                   LPSD0080
C                N        - NUMBER OF COLUMNS IN MATRIX A                LPSD0090
C                IA       - ROW DIMENSION OF A AND AINV IN CALLING       LPSD0100
C                             PROGRAM (IA MUST BE GREATER THAN OR EQUAL   LPSD0110
C                             TO M).                                     LPSD0120
C                AINV     - THE TRANSPOSE OF THE PSEUDO-INVERSE OF A     LPSD0130
C                             IS STORED IN THE M X N MATRIX AINV.        LPSD0140
C                IDGT     - THE ELEMENTS OF A ARE ASSUMED TO BE CORRECT  LPSD0150
C                             TO IDGT DECIMAL PLACES. IDGT IS AN INPUT   LPSD0160
C                             PARAMETER.                                 LPSD0170
C                WKAREA   - WORK AREA OF DIMENSION AT LEAST EQUAL TO     LPSD0180
C                             N(N+4).                                    LPSD0190
C                IER      - ERROR PARAMETER                              LPSD0200
C                           TERMINAL ERROR = 128 + N                     LPSD0210
C                             N = 1 INDICATES THAT A IS THE ZERO MATRIX  LPSD0220
C                               TO IDGT ACCURACY(IN NORM).               LPSD0230
C   PRECISION             - SINGLE                                       LPSD0240
C   REQD. IMSL ROUTINES   - VSORTM,LSVALR,UERTST                         LPSD0250
C   LANGUAGE              - FORTRAN                                      LPSD0260
C--------------------------------------------------------------------------LPSD0270
```

CALL LPSDOR (A,M,N,IA,AINV,IDGT,WKAREA,IER)

Purpose

LPSDOR calculates the pseudo-inverse of the M by N matrix A with M not less than N.

Algorithm

The routine calculates the singular value decomposition of A by calling IMSL routine LSVALR. $A=USV^T$, where U is an M by N matrix with orthonormalized columns, S is an N by N diagonal matrix containing the singular values of A, and V is an N by N orthogonal matrix. The pseudo-inverse is given by $V(S^{-1})U^T$.

Programming Notes

1. In the main program, both A and AINV are assumed to be of dimension M by N (IA not less than M). Thus the transpose of the pseudo-inverse of A is stored in AINV. AINV may occupy the same storage as does A, or completely separate storage.

2. The work area, WKAREA, must be a vector of length at least N(N+4).

3. If A is the zero matrix, a terminal error message is printed, and the zero pseudo-inverse is not stored in AINV. Instead it will contain U.

Accuracy

The elements of A are assumed correct to IDGT significant digits. The routine computes the exact pseudo-inverse, without any roundoff error, to a matrix $\bar{A}$ which differs by A in the Euclidean norm by less than 10^{-IDGT}, and which has minimal rank.

APPENDIX E

INTERP DESCRIPTION

Identification:

INTERP Aitken's Kth Degree Polynomial Interpolation of Tabular Data
CDC 6600 FORTRAN Subroutine Subprogram

Purpose:

Given an ascending or descending table X(I) of independent variables and a corresponding table Y(I) of dependent variables, subroutine INTERP computes a Kth degree polynomial interpolated value YO for a given abscissa XO. If the argument falls outside the range of tabular values, extrapolation is performed.

Control:

```
DIMENSION X(N), Y(N), TEMP(2(K+1))
CALL INTERP (X,Y,N,K,XO,YO,TEMP,IER)
```

where:

X(I)	List of values of the independent variable in either ascending or descending order
Y(I)	List of the corresponding values of the dependent variable
N	Number of X, Y pairs
K	Degree of the interpolating polynomial (K less than N)
XO	Point at which interpolation is requested
YO	The computed interpolated value
TEMP	A one-dimensional array of 2*(K+1) words of temporary storage
IER	= 0 Interpolation successfully performed = 1 Extrapolation successfully performed = 2 Either degree of interpolation out of range or two identical independent variables.

Method:

Repeated bisection of the index (subscript) of the list of values of X is carried out until the argument is isolated between two consecutive values. Then Aitken's recursion of linear interpolation is performed using an equal number of points on either side of XO and, for even degrees, one additional point nearest XO.

Remarks:

The user is cautioned that increasing the degree of the fitted polynomial does not necessarily increase the "accuracy" of the interpolation. Indeed, this may introduce spurious oscillations into the interpolated values since the higher the degree of a polynomial, the more relative maxima and minima it can have. In particular, if the tabular data involves appreciable "noise", then use of a technique which "smooths" rather than one which "fits" (as does INTERP) may be more appropriate.

The user is also encouraged to test to see if IER = 2 if there is any question concerning the uniqueness of the independent variables.

Storage:

INTERP uses 254_8 words of storage and no common.

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